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Pre-Leaving Certificate Examination, 2018

Mathematics

Higher Level

Marking Scheme

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Pre-Leaving Certificate Examination, 2018

Mathematics

Higher Level – Paper 1
Marking Scheme (300 marks)

Structure of the Marking Scheme

Students' responses are marked according to different scales, depending on the types of response anticipated.
Scales labelled A divide students' responses into two categories (correct and incorrect).
Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on.
These scales and the marks that they generate are summarised in the following table:

Scale label	A	B	C	D
No. of categories	2	3	4	5
5 mark scale		**0, 2, 5**	**0, 2, 4, 5**	**0, 2, 3, 4, 5**
10 mark scale			**0, 4, 7, 10**	**0, 4, 6, 8, 10**
15 mark scale				**0, 5, 9, 12, 15**

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)
- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)
- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (mid partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving ❶ incorrect rounding, ❷ omission of units, ❸ a misreading that does not oversimplify the work or ❹ an arithmetical error that does not oversimplify the work, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk.
Thus, for example, scale **10C*** indicates that 9 marks may be awarded.

- The * for units to be applied only if the student's answer is fully correct.
- The * to be applied once only **within each section (a), (b), (c), *etc.*** of all questions.
- The * penalty is not applied for the omission of units in currency solutions.

Unless otherwise specified, accept correct answer with or without work shown.

Accept students' work in one part of a question for use in subsequent parts of the question, unless this oversimplifies the work involved.

Summary of Marks – 2018 LC Maths (Higher Level, Paper 1)

Section A

Q.1	**(a)**		15D (0, 5, 9, 12, 15)
	(b)		10D (0, 4, 6, 8, 10)
			25

Q.2	**(a)**	**(i)**	10C (0, 4, 7, 10)
		(ii)	10D (0, 4, 6, 8, 10)
	(b)		5C (0, 2, 4, 5)
			25

Q.3	**(a)**		10D (0, 4, 6, 8, 10)
	(b)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
			25

Q.4	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	10D* (0, 4, 6, 8, 10)
	(b)		10D* (0, 4, 6, 8, 10)
			25

Q.5	**(a)**		10D (0, 4, 6, 8, 10)
	(b)		5C (0, 2, 4, 5)
	(c)		10D (0, 4, 6, 8, 10)
			25

Q.6	**(a)**	**(i)**	10C (0, 4, 7, 10)
		(ii)	5C (0, 2, 4, 5)
	(b)		10D (0, 4, 6, 8, 10)
			25

Section B

Q.7	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C* (0, 2, 4, 5)
		(iii)	10D* (0, 4, 6, 8, 10)
		(iv)	5C (0, 2, 4, 5)
	(b)	**(i)**	5B (0, 2, 5)
		(ii)	5B (0, 2, 5)
		(iii)	10D* (0, 4, 6, 8, 10)
			45

Q.8	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	10D (0, 4, 6, 8, 10)
		(iii)	10D* (0, 4, 6, 8, 10)
		(iv)	5C* (0, 2, 4, 5)
	(b)	**(i)**	10D* (0, 4, 6, 8, 10)
		(ii)	10C (0, 4, 7, 10)
		(iii)	5D (0, 2, 3, 4, 5)
			55

Q.9	**(a)**	**(i)**	5B (0, 2, 5)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
		(iv)	5C (0, 2, 4, 5)
	(b)	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
		(iii)	10D (0, 4, 6, 8, 10)
		(iv)	5C (0, 2, 4, 5)
		(v)	5C (0, 2, 4, 5)
			50

Current Marking Scheme

Assumptions about these marking schemes on the basis of past SEC marking schemes should be avoided. While the underlying assessment principles remain the same, the exact details of the marking of a particular type of question may vary from a similar question asked by the SEC in previous years in accordance with the contribution of that question to the overall examination in the current year. In setting these marking schemes, we have strived to determine how best to ensure the fair and accurate assessment of students' work and to ensure consistency in the standard of assessment from year to year. Therefore, aspects of the structure, detail and application of the marking schemes for these examinations are subject to change from past SEC marking schemes and from one year to the next without notice.

Pre-Leaving Certificate Examination, 2018

Mathematics

Higher Level – Paper 1
Marking Scheme (300 marks)

General Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer **all nine** questions.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

Answers should be given in simplest form, where relevant.

Section A	**Concepts and Skills**	**150 marks**

Answer **all six** questions from this section.

Question 1 **(25 marks)**

1(a) Solve the simultaneous equations: **(15D)**

$$2x + \frac{y}{2} - 2z = 8 \quad (\times 2)$$
$$\frac{x}{2} + \frac{y}{3} + \frac{5z}{9} = 0 \quad (\times 18)$$
$$\frac{x}{6} - \frac{y}{4} + \frac{z}{12} = \frac{7}{8} \quad (\times 24)$$

$$\Rightarrow \quad 4x + y - 4z = 16 \quad \ldots ①$$
$$9x + 6y + 10z = 0 \quad \ldots ②$$
$$4x - 6y + 2z = 21 \quad \ldots ③$$

Equating ② and ③:

② $9x + 6y + 10z = 0$

③ $4x - 6y + 2z = 21$

$13x + 12z = 21$ ... ④

Equating ① and ③:

① $4x + y - 4z = 16 \quad (\times 6)$

③ $4x - 6y + 2z = 21 \quad (\times 1)$

$\Rightarrow$ $24x + 6y - 24z = 96$

$4x - 6y + 2z = 21$

$28x - 22z = 117$ ... ⑤

Question 1 (cont'd.)

1(a) (cont'd.)

Equating ④ and ⑤:

④ $13x + 12z = 21 \quad (\times 11)$
⑤ $28x - 22z = 117 \quad (\times 6)$

$$\Rightarrow \begin{aligned} 143x + 132z &= 231 \\ \underline{168x - 132z} &= \underline{702} \\ 311x &= 933 \end{aligned}$$

$\Rightarrow x = 3$

Substituting into ④:

④ $13x + 12z = 21$

$\Rightarrow 13(3) + 12z = 21$

$\Rightarrow 12z = 21 - 39 = -18$

$\Rightarrow z = -\frac{3}{2}$

Substituting into ①:

① $4x + y - 4z = 16$

$\Rightarrow 4(3) + y - 4(-\frac{3}{2}) = 16$

$\Rightarrow y = 16 - 12 - 6 = -2$

Scale 15D (0, 5, 9, 12, 15)

Low partial credit: (5 marks)	–	Any relevant first step, *e.g.* eliminates fractions in at least one equation.
Mid partial credit: (9 marks)	–	Finds correctly one equation with two variables, *e.g.* $13x + 12z = 21$.
High partial credit: (12 marks)	– –	Finds correctly two equations with the <u>same</u> two variables, *e.g.* $13x + 12z = 21$ and $28x - 22z = 117$, but fails to finish <u>or</u> finishes incorrectly. Finds one variable (x, y <u>or</u> z) only.

Question 1 (cont'd.)

1(b) If $(x + a)^2$ is a factor of $10x^3 + 21ax^2 + 20abx + 25a$, where a and b are non-zero constants, find the possible values of a and b. **(10D)**

❶ $(x + a)^2 = x^2 + 2ax + a^2$

$$\begin{array}{r|l} & 10x + a \\ \hline x^2 + 2ax + a^2 & 10x^3 + 21ax^2 + 20abx \qquad + 25a \\ & \underline{-10x^3 - 20ax^2 - 10a^2x} \\ & \qquad ax^2 + x(20ab - 10a^2) + 25a \\ & \qquad \underline{-ax^2 - 2a^2x \qquad - a^3} \\ & \qquad\qquad\qquad 0 \end{array}$$

as $(x + a)^2$ is a factor, $\Rightarrow$ remainder = 0

$\Rightarrow a^3 = 25a$

$\Rightarrow a^2 = 25$

$\Rightarrow a = 5$ <u>or</u> -5

also $2a^2 = 20ab - 10a^2$

$\Rightarrow 2a = 20b - 10a$

$\Rightarrow 20b = 12a$

$\Rightarrow b = \frac{3}{5}a$

$\Rightarrow b = \frac{3}{5}(5) = 3$ $\qquad b = \frac{3}{5}(-5) = -3$

<u>or</u>

❷ $(x + a)^2 = x^2 + 2ax + a^2$

$\Rightarrow (x^2 + 2ax + a^2)(10x + c) = 10x^3 + 21ax^2 + 20abx + 25a$

$\Rightarrow 10x^3 + 20ax^2 + 10a^2x + cx^2 + 2acx + a^2c = 10x^3 + 21ax^2 + 20abx + 25a$

Comparing terms:

x^2: $20a + c = 21a$

$\Rightarrow 21a - 20a = c$

$\Rightarrow a = c$

x: $10a^2 + 2ac = 20ab$

$\Rightarrow 10a^2 + 2a(a) = 20ab$

$\Rightarrow 10a + 2a = 20b$

$\Rightarrow 20b = 12a$

$\Rightarrow b = \frac{3}{5}a$

constants: $a^2c = 25a$

$\Rightarrow a^2(a) = 25a$

$\Rightarrow a^2 = 25$

$\Rightarrow a = 5$ <u>or</u> -5

$\Rightarrow b = \frac{3}{5}(5) = 3$ $\qquad b = \frac{3}{5}(-5) = -3$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant correct step, *e.g.* some correct division in dividing $x^2 + 2ax + a^2$ into equation <u>or</u> some correct multiplication of $(x^2 + 2ax + a^2)(10x + c)$.
	–	Writes down $2x - 1$ is a factor of equation and attempts to divide.
Mid partial credit: (6 marks)	–	Fully correct division <u>or</u> multiplication, but fails to progress.
High partial credit: (8 marks)	–	Finds $a = \pm 5$ <u>or</u> $b = \frac{3}{5}a$, but fails to find a and b (both values) [Method ❶].
	–	Finds $a = c$ <u>and</u> $b = \frac{3}{5}a$, but fails to find a and b (both values) [Method ❷].

Question 2 **(25 marks)**

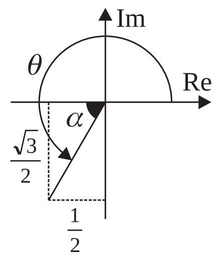
2(a) $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is a complex number, where $i^2 = -1$.

(i) Write z in polar form. **(10C)**

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}i = r(\cos\theta + i\sin\theta)$$

❶
$$\begin{aligned} r &= |z| \\ &= \sqrt{(-\tfrac{1}{2})^2 + (-\tfrac{\sqrt{3}}{2})^2} \\ &= \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

❷
$$\begin{aligned} \tan\alpha &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} \\ &= \sqrt{3} \\ \Rightarrow \alpha &= \tan^{-1}\sqrt{3} \\ &= \frac{\pi}{3} \text{ or } 60° \end{aligned}$$



$$\Rightarrow \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$ <u>or</u> $\theta = 180° + 60° = 240°$

$$\begin{aligned} -\frac{1}{2} - \frac{\sqrt{3}}{2}i &= r(\cos\theta + i\sin\theta) \\ &= 1(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) \\ &= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \end{aligned}$$

<u>or</u> $\cos 240° + i\sin 240°$

<u>or</u> $\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})$

<u>or</u> $\cos(-120°) + i\sin(-120°)$

Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down relevant formula to express a complex number in polar form.
	–	Finds correct r <u>or</u> α (reference angle).
	–	Plots z correctly on an Argand diagram.
High partial credit: (7 marks)	–	Finds correct values for both r <u>and</u> θ, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds correct values for both r <u>and</u> α (reference angle), but complex number in wrong quadrant and finishes correctly.

Question 2 (cont'd.)

2(a) (cont'd.)

(ii) Hence, find the four complex numbers w such that $w^4 = z$.
Give your answers in rectangular form. **(10D)**

$$w^4 = z$$
$$= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$
$$= \cos\left(\frac{4\pi}{3} + 2n\pi\right) + i\sin\left(\frac{4\pi}{3} + 2n\pi\right)$$
$$\Rightarrow \quad w = \left[\cos\left(\frac{4\pi}{3} + 2n\pi\right) + i\sin\left(\frac{4\pi}{3} + 2n\pi\right)\right]^{\frac{1}{4}}$$
$$= \cos\left(\frac{4\pi}{12} + \frac{2n\pi}{4}\right) + i\sin\left(\frac{4\pi}{12} + \frac{2n\pi}{4}\right)$$
$$= \cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{n\pi}{2}\right)$$

For $n = 0$

$$w_1 = \cos\left(\frac{\pi}{3} + \frac{(0)\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{(0)\pi}{2}\right)$$
$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

For $n = 1$

$$w_2 = \cos\left(\frac{\pi}{3} + \frac{(1)\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{(1)\pi}{2}\right)$$
$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

For $n = 2$

$$w_3 = \cos\left(\frac{\pi}{3} + \frac{(2)\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{(2)\pi}{2}\right)$$
$$= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$
$$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

For $n = 3$

$$w_4 = \cos\left(\frac{\pi}{3} + \frac{(3)\pi}{2}\right) + i\sin\left(\frac{\pi}{3} + \frac{(3)\pi}{2}\right)$$
$$= \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $w = \sqrt[4]{z}$ or $z^{\frac{1}{4}}$ or $w^4 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ or similar.
	–	Writes down z in general polar form, *i.e.* $z = \cos\left(\frac{4\pi}{3} + 2n\pi\right) + i\sin\left(\frac{4\pi}{3} + 2n\pi\right)$.

Question 2 (cont'd.)

2(a) (ii) (cont'd.)

Mid partial credit: (6 marks)	–	De Moivre's Theorem applied correctly with $n = \frac{1}{4}$, but fails to progress, *e.g.* $w = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and stops.
	–	Finds correct general term for w, but fails to substitute $n = 0, 1, 2, 3$ into expression, *i.e.* $w = \cos(\frac{\pi}{3} + \frac{n\pi}{2}) + i\sin(\frac{\pi}{3} + \frac{n\pi}{2})$ and stops or continues incorrectly.
High partial credit: (8 marks)	–	Finds first root, $w_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ or $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ correctly from general polar form, but fails to find or finds incorrect other roots.
	–	Finds all roots in polar form, but fails to convert or converts incorrectly to rectangular form.
	–	Substantive work towards finding all four roots with one error/omission.

2(b) Use De Moivre's Theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. **(5C)**

Consider $(\cos\theta + i\sin\theta)^3$

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ ... De Moivre's Theorem

Expanding $(\cos\theta + i\sin\theta)^3$

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3(\cos\theta)(i\sin\theta)^2 + (i\sin\theta)^3$$
$$= [\cos^3\theta - 3\cos\theta\sin^2\theta] + i[3\cos^2\theta\sin\theta - \sin^3\theta]$$

Equating the imaginary parts

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ and stops.
	–	Expands $(\cos\theta + i\sin\theta)^3$ correctly and stops.
High partial credit: (4 marks)	–	Finds both expansions for $(\cos\theta + i\sin\theta)^3$ correctly and equates imaginary parts, but fails to finish or finishes incorrectly.
	–	Substantive work towards finding $\sin 3\theta$, but with one error/omission.

Question 3 **(25 marks)**

3(a) Solve the equation $3^{2x+2} - 28(3^x) + 3 = 0$. [Hint: Let $y = 3^x$.] **(10D)**

Let $y = 3^x$

$$\Rightarrow 3^{2x+2} = 3^{x+x+2} = (3^x)(3^x)(3^2) = (y)(y)(9) = 9y^2$$

$$3^{2x+2} - 28(3^x) + 3 = 0$$
$$\Rightarrow 9y^2 - 28y + 3 = 0$$
$$\Rightarrow (9y-1)(y-3) = 0$$

$$\Rightarrow 9y - 1 = 0 \qquad \Rightarrow y - 3 = 0$$
$$\Rightarrow 9y = 1 \qquad \Rightarrow y = 3$$
$$\Rightarrow y = \frac{1}{9}$$

$$\Rightarrow y = 3^x = \frac{1}{9} \qquad \Rightarrow y = 3^x = 3$$
$$\Rightarrow 3^x = \frac{1}{9} = 3^{-2} \qquad \Rightarrow 3^x = 3 = 3^1$$
$$\Rightarrow x = -2 \qquad \Rightarrow x = 1$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	– –	Any relevant first step, *e.g.* writes down $3^{2x+2} = (3^{2x})(3^2)$, $(3^x)(3^x)(3^2)$ <u>or similar</u>. Finds $3^{2x+2} = 9y^2$ correctly <u>and stops</u> <u>or</u> continues incorrectly.
Mid partial credit: (6 marks)	–	Substitutes y correctly into quadratic eqn. <u>and</u> finds correct factors of equation [ans. $(9y-1)(y-3) = 0$].
High partial credit: (8 marks)	– –	Finds two correct values for y, but fails to find both corresponding values for x <u>or</u> finds incorrect values for x. Finds one correct value for y and finds correct corresponding value for x.

Question 3 (cont'd.)

3(b) (i) Prove by induction that the sum of the squares of the first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots + n^2$, is $\frac{n(n+1)(2n+1)}{6}$. **(10D)**

❶ P(n):

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

❷ P(1):

Test hypothesis for $n = 1$

$$1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$$

$\Rightarrow$ True for $n = 1$

❸ P(k):

Assume hypothesis for $n = k$ is true

$$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

❹ P($k + 1$):

Test hypothesis for $n = k + 1$

To Prove:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

<u>Proof:</u>

Consider LHS:

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \text{RHS} \end{aligned}$$

$\Rightarrow$ True for $n = k + 1$

So, P(k + 1) is true whenever P(k) is true.

Since P(1) is true, then by induction P(n) is true for any positive integer n / all $n \in \mathbb{N}$.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correctly P(1) step <u>and stops</u>.
Mid partial credit: (6 marks)	–	Writes down correctly P(1) <u>and</u> P(k) <u>or</u> P(k + 1) steps.
High partial credit: (8 marks)	– –	Writes down correctly P(1) step <u>and</u> P(k) and uses P(k) to prove P(k + 1) step, but fails to finish <u>or</u> finishes incorrectly. Writes down all steps correctly, but no conclusion <u>or</u> incorrect conclusion given.

Question 3 (cont'd.)

3(b) (cont'd.)

(ii) Hence, or otherwise, evaluate the sum of the squares of all the natural numbers from 30 to 60, inclusive. **(5C)**

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \quad 30^2 + 31^2 + 33^2 + \ldots + 60^2 = S_{60} - S_{29}$$

$$= \frac{60(60+1)(120+1)}{6} - \frac{29(29+1)(58+1)}{6}$$

$$= \frac{60(61)(121)}{6} - \frac{29(30)(59)}{6}$$

$$= \frac{442{,}860}{6} - \frac{51{,}330}{6}$$

$$= 73{,}810 - 8{,}555$$

$$= 65{,}255$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down 'sum = $S_{60} - S_{29}$' <u>or similar</u>. [Do not accept 'sum = $S_{60} - S_{30}$'.]
	–	Correct substitution into formula from part (b)(i) using n = 29, 30 <u>or</u> 60.
High partial credit: (4 marks)	–	Correct substitution into formula using n = 29 <u>and</u> 60, but fails to evaluate <u>or</u> evaluates incorrectly.
	–	Incorrect substitution into formula using n = 30, but otherwise finishes correctly [ans. 73,810 – 9,455 = 64,355].

Question 4 **(25 marks)**

Dan and Kate plan to buy a house which costs €250,000. In order to get a mortgage on the property, the couple need to save a deposit of 10% of the purchase price. They open a savings account in their local Credit Union which offers an annual equivalent rate (AER) of 3·5%.

4(a) **(i)** Show that the rate of interest, compounded monthly, which is equivalent to an AER of 3·5% is 0·287%, correct to three decimal places. **(5C)**

$$r = \text{annual equivalent rate (AER)}$$
$$i = \text{monthly percentage rate}$$

$$\begin{aligned}
F &= P(1 + r) \\
&= P(1 + i)^t \\
\Rightarrow 1(1 + r) &= 1(1 + i)^t \\
\Rightarrow 1(1 + 0{\cdot}035) &= 1(1 + i)^{12} \\
\Rightarrow 1{\cdot}035 &= (1 + i)^{12} \\
\Rightarrow 1 + i &= (1{\cdot}035)^{\frac{1}{12}} \\
\Rightarrow i &= 1{\cdot}002870898... - 1 \\
&= 0{\cdot}002870898... \\
\Rightarrow r &= 0{\cdot}2870898...\% \\
&\cong 0{\cdot}287\%
\end{aligned}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula $F = P(1 + i)^t$ <u>and stops</u>.
	–	Some correct substitution into correct formula (not stated) <u>and stops</u> <u>or</u> continues.
	–	Correct substitution into incorrect formula <u>and stops</u> <u>or</u> continues.
High partial credit: (4 marks)	–	Fully correct substitution into formula, *i.e.* $1(1 + 0{\cdot}035) = 1(1 + i)^{12}$ <u>or equivalent</u>, but fails to find <u>or</u> finds incorrect rate.
	–	Final answer not given as a percentage, *i.e.* $r = 0{\cdot}002870898...$

4(a) (cont'd.)

(ii) Dan and Kate decide to put €500 in the savings account at the beginning of each month. How long will it take them to save up the deposit for the house? Give your answer in months, correct to the nearest month. **(10D*)**

Deposit required $= 10\%$ of 250,000

$= 250{,}000 \times \frac{10}{100}$

$=$ €25,000

Value of savings instalments after n months

$F = P(1 + i)^t$

$= 500(1 + 0{\cdot}00287)^n$

$= 500(1{\cdot}00287)^n$

Month	Instalment (€)	Value of instalment after n months
1	500	$500(1{\cdot}00287)^n$
2	500	$500(1{\cdot}00287)^{n-1}$
3	500	$500(1{\cdot}00287)^{n-2}$
...	...	...
t	500	$500(1{\cdot}00287)^1$

$\Rightarrow$ Geometric series with $a = 500(1{\cdot}00287)$ and $r = 1{\cdot}00287$

$S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow$ $25{,}000 = \frac{500(1{\cdot}00287)(1 - 1{\cdot}00287^n)}{1 - 1{\cdot}00287}$

$= -174{,}716{\cdot}027874...(1 - 1{\cdot}00287^n)$

$= 174{,}716{\cdot}027874...(1{\cdot}00287^n - 1)$

$\Rightarrow$ $1{\cdot}00287^n - 1 = \frac{25{,}000}{174{,}716{\cdot}027874...}$

$= 0{\cdot}143089333...$

$\Rightarrow$ $1{\cdot}00287^n = 0{\cdot}143089333... + 1$

$= 1{\cdot}143089333...$

$\Rightarrow$ $n = \log_{1{\cdot}00287}(1{\cdot}143089333...)$

$= 46{\cdot}664235...$

$\cong 47$ months

$\Rightarrow$ it will take Dan and Kate 47 months to save up the deposit

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* reference to value of first instalment or subsequent instalments after n months $= 500(1{\cdot}00287)^n$, where $1 < n \leq 60$.
	–	Finds deposit required [ans. 25,000].
	–	Recognises value of savings instalments after n months as a sum of a GP with some correct substitution into S_n formula.
Mid partial credit: (6 marks)	–	Fully correct substitution into S_n formula, but fails to progress.
High partial credit: (8 marks)	–	Substantive work towards finding value of n with one error/omission or equation in n (n no longer an index).

* Deduct 1 mark off correct answer only if not rounded or incorrectly rounded - apply only once to each section (a), (b), (c), *etc*. of question.

* No deduction applied for the omission of or incorrect use of units ('months').

4(b) After saving for three years, Dan and Kate find the perfect house. They decide to borrow the remainder of the **deposit** at a monthly interest rate of 0·425%, fixed for the term of the loan. The loan is to be repaid in equal monthly repayments over five years and the first repayment is due one month after the loan is issued. Calculate the amount of each monthly repayment, correct to the nearest cent. **(10D*)**

❶ Value of savings instalments after 36 months

instalments = 3 × 12 = 36

$$F = P(1+i)^t = 500(1+0{\cdot}00287)^n = 500(1{\cdot}00287)^n$$

Month	Instalment (€)	Value of instalment after 36 months
1	500	$500(1{\cdot}00287)^{36}$
2	500	$500(1{\cdot}00287)^{35}$
3	500	$500(1{\cdot}00287)^{34}$
...	...	...
36	500	$500(1{\cdot}00287)^{1}$

⇒ Geometric series with $n = 36$, $a = 500(1{\cdot}00287)$ and $r = 1{\cdot}00287$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_{36} = \frac{500(1{\cdot}00287)(1-1{\cdot}00287^{36})}{1-1{\cdot}00287} = 18{,}988{\cdot}506021... \cong €18{,}988{\cdot}51$$

❷ Remainder of deposit:

Remainder = 25,000 – 18,988·51 = €6,011·49

❸ Monthly repayments:

① Sum of geometric series:

repayments = 12 × 5 = 60

$$F = P(1+i)^t$$

$$\Rightarrow P = \frac{F}{(1+i)^t}$$

i = 0·00425

X = fixed monthly repayment

$$\Rightarrow P = \frac{X}{(1+0{\cdot}00425)^t} = \frac{X}{1{\cdot}00425^t}$$

Month	Present value of future repayment (P)	Future repayment (F)
1	$\frac{X}{1{\cdot}00425^1}$	X
2	$\frac{X}{1{\cdot}00425^2}$	X
...	...	...
60	$\frac{X}{1{\cdot}00425^{60}}$	X

Question 4 (cont'd.)

4(b) (cont'd.)

$\Rightarrow$ Geometric series with $n = 60$, $a = \dfrac{X}{1{\cdot}00425}$ and $r = \dfrac{1}{1{\cdot}00425}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_{60} = \frac{\dfrac{X}{1\cdot00425}\left(1-\dfrac{1}{1\cdot00425^{60}}\right)}{1-\dfrac{1}{1\cdot00425}}$$

$$= \frac{X(0{\cdot}223713...)}{0{\cdot}004232...}$$

$$= 52{\cdot}862274...X$$

$$= 6{,}011{\cdot}49$$

$\Rightarrow 52{\cdot}862274...X = 6{,}011{\cdot}49$

$\Rightarrow X = 113{\cdot}719851...$

$\cong$ €113·72

<u>or</u>

① <u>Amortisation:</u>

$$A = P\frac{i(1+i)^t}{(1+i)^t-1}$$

$t = 12 \times 5$

$= 60$ months

$i = 0{\cdot}00425$

$P = 6{,}011{\cdot}49$

$X =$ fixed monthly repayment

$$\Rightarrow X = \frac{6{,}011{\cdot}49(0{\cdot}00425)(1+0{\cdot}00425)^{60}}{(1+0{\cdot}00425)^{60}-1}$$

$$= \frac{6{,}011{\cdot}49(0{\cdot}00425)(1{\cdot}00425)^{60}}{(1{\cdot}00425)^{60}-1}$$

$= 113{\cdot}719851...$

$\cong$ €113·72

Scale 10D* (0, 4, 6, 8, 10)

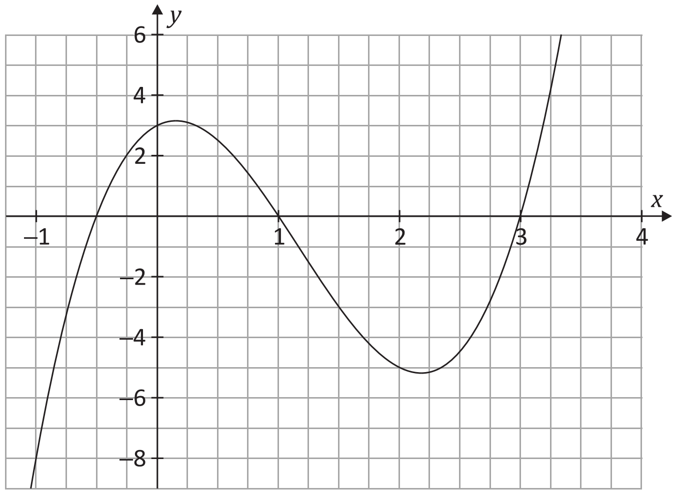
Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* finds correct # instalments [ans. 3 × 12 = 36] <u>and/or</u> # repayments [ans. 5 × 12 = 60].
	–	Recognises value of savings instalments after 36 months as a sum of a GP with some correct substitution into S_n formula.
Mid partial credit: (6 marks)	–	Finds correct value of savings instalments after 36 months (S_{36}) [ans. €18,988·51] <u>or</u> remainder of deposit [€6,011·49].
	–	Recognises sum of future repayments as a sum of a GP with some correct substitution into S_n formula.
	–	Writes down correct relevant formula for amortisation with some correct substitution into formula.
High partial credit: (8 marks)	–	Fully correct substitution into S_n <u>or</u> amortisation formula, but fails to finish <u>or</u> finishes incorrectly.

* Deduct 1 mark off correct answer only if not rounded <u>or</u> incorrectly rounded - apply only once to each section (a), (b), (c), *etc*. of question.

* No deduction applied for the omission of <u>or</u> incorrect use of units in questions involving currency.

Question 5 **(25 marks)**

The diagram shows part of the graph of a cubic function $f(x)$, where $x \in \mathbb{R}$.



5(a) Find the equation of $f(x)$. **(10D)**

From the graph, the roots of $f(x)$ are
$x = -0{\cdot}5$, $x = 1$ and $x = 3$

$\Rightarrow \quad f(x) = k(x + 0{\cdot}5)(x - 1)(x - 3)$
$= k(x + 0{\cdot}5)(x^2 - 4x + 3)$
$= k(x^3 - 4x^2 + 3x + 0{\cdot}5x^2 - 2x + 1{\cdot}5)$
$= k(x^3 - 3{\cdot}5x^2 + x + 1{\cdot}5)$

From the graph, $f(0) = 3$
$\Rightarrow \quad f(x) = k(x^3 - 3{\cdot}5x^2 + x + 1{\cdot}5)$
$\Rightarrow \quad f(0) = k(0^3 - 3{\cdot}5(0)^2 + 0 + 1{\cdot}5)$
$= 3$
$\Rightarrow \quad k(1{\cdot}5) = 3$
$\Rightarrow \quad k = 2$

$\Rightarrow \quad f(x) = 2(x^3 - 3{\cdot}5x^2 + x + 1{\cdot}5)$
$= 2x^3 - 7x^2 + 2x + 3$

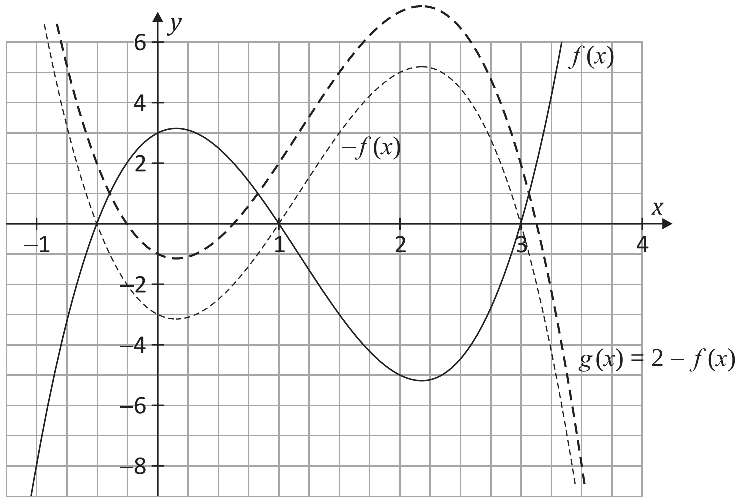
Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down all three correct roots of $f(x)$, *i.e.* $x = -0{\cdot}5$, $x = 1$ and $x = 3$ <u>and stops</u>.
	–	Finds at least two correct factors of $f(x)$, *i.e.* $x + 0{\cdot}5$ <u>or</u> $2x + 1$, $x - 1$ and $x - 3$.
	–	Uses graph to find $f(0) = 3$.
Mid partial credit: (6 marks)	–	Finds $f(x) = k(x + 0{\cdot}5)(x - 1)(x - 3)$, but fails to progress.
High partial credit: (8 marks)	–	Finds $f(x) = k(x^3 - 3{\cdot}5x^2 + x + 1{\cdot}5)$, but fails to find correct value of k.
	–	Finds $f(x) = 2x^3 - 7x^2 + 2x + 3$ without reference to $f(0) = 3$.

Question 5 (cont'd.)

5(b) On the diagram above, draw the graph of the function $g(x) = 2 - f(x)$, where $x \in \mathbb{R}$. **(5C)**

❶ Graphical



or

❷ Calculation

⇒ $f(x)$	=	$2x^3 - 7x^2 + 2x + 3$	... answer from part (b)
$g(x)$	=	$2 - f(x)$	
	=	$2 - (2x^3 - 7x^2 + 2x + 3)$	
	=	$2 - 2x^3 + 7x^2 - 2x - 3$	
	=	$-2x^3 + 7x^2 - 2x - 1$	

Substituting values into $g(x)$

Points = (–1, 10), (0, –1), (1, 2), (2, 7), (3, 2), (4, –25)

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* draws graph or sketches graph of $-f(x)$ on diagram.
	–	Finds correct equation of $g(x) = 2 - f(x)$, *i.e.* $g(x) = -2x^3 + 7x^2 - 2x - 1$.
High partial credit: (4 marks)	–	Draws correct shape of $g(x)$, but graph does not pass clearly through (–0·5, 2), (1, 2), (3, 2) and (0, –1).

Question 5 (cont'd.)

5(c) Use integration to find the average value of $g(x)$ over the interval $0 \le x \le 3$, $x \in \mathbb{R}$. **(10D)**

Average value of $g(x)$ in the interval $[a, b]$

$$= \frac{1}{b-a}\int_a^b g(x)\,dx$$

$$\begin{aligned} g(x) &= 2 - f(x) \\ &= 2 - [2x^3 - 7x^2 + 2x + 3] \\ &= -2x^3 + 7x^2 - 2x + 2 - 3 \\ &= -2x^3 + 7x^2 - 2x - 1 \end{aligned}$$

$\Rightarrow$ Average value of $g(x)$

$$\begin{aligned} &= \frac{1}{3-0}\int_0^3 (-2x^3 + 7x^2 - 2x - 1)\,dx \\ &= \frac{1}{3}\left[-2\frac{x^4}{4} + 7\frac{x^3}{3} - 2\frac{x^2}{2} - x\right]\Big|_0^3 \\ &= \frac{1}{3}\left[-\frac{2}{4}(3)^4 + \frac{7}{3}(3)^3 - \frac{2}{2}(3)^2 - 3\right] \\ &= \frac{1}{3}\left[-\frac{81}{2} + 63 - 9 - 3\right] \\ &= \frac{1}{3}[-40\cdot5 + 51] \\ &= \frac{1}{3}[10\cdot5] \\ &= 3\cdot5 \end{aligned}$$

** Accept students' answers for $f(x)$ from part (a) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down relevant formula for the average value of a function.
	–	Formulates integral (with correct limits), *i.e.* $\frac{1}{3}\int_0^3 (-2x^3 + 7x^2 - 2x - 1)\,dx$.
	–	Integrates one term correctly.
Mid partial credit: (6 marks)	–	Integrates all terms correctly, but omits $\frac{1}{b-a}$ from formula for average value <u>and</u> attempts to evaluate.
	–	Correct integration to find average value of $g(x)$, *i.e.* $\frac{1}{3}\left[-2\frac{x^4}{4} + 7\frac{x^3}{3} - 2\frac{x^2}{2} - x\right]\Big|_0^3$, but fails to evaluate <u>or</u> evaluates incorrectly <u>or</u> evaluates using incorrect limits.
High partial credit: (8 marks)	–	Correct integration to find average value of $g(x)$ with full substitution of limits, *i.e.* $\frac{1}{3}\left[-\frac{2}{4}(3)^4 + \frac{7}{3}(3)^3 - \frac{2}{2}(3)^2 - 3\right]$ <u>or similar</u>, but fails to evaluate <u>or</u> evaluates incorrectly.

Question 6 **(25 marks)**

6(a) Let $f(x) = \ln\sqrt{\dfrac{x+1}{x-1}}$, for $x > 1$, where $x \in \mathbb{R}$.

(i) Use the rules of logarithms to find $f'(x)$, the derivative of $f(x)$.
Give your answer in the form $\dfrac{a}{a - ax^2}$, where $a \in \mathbb{Z}$. **(10C)**

$$f(x) = \ln\sqrt{\frac{x+1}{x-1}}$$
$$= \ln\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}$$
$$= \frac{1}{2}[\ln(x+1) - \ln(x-1)]$$
$$\Rightarrow \quad f'(x) = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right]$$
$$= \frac{x-1-(x+1)}{2(x+1)(x-1)}$$
$$= \frac{-2}{2(x^2-1)}$$
$$= \frac{-1}{x^2-1}$$
$$= \frac{1}{1-x^2}$$

Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* uses rules of logarithms to find $\ln\left(\dfrac{x+1}{x-1}\right)^{\frac{1}{2}}$ or $\dfrac{1}{2}[\ln(x+1) - \ln(x-1)]$.
	–	Differentiates one term correctly.
High partial credit: (7 marks)		Differentiates $f(x)$ correctly, *i.e.* $f'(x) = \dfrac{1}{2}\left[\dfrac{1}{x+1} - \dfrac{1}{x-1}\right]$, but fails to find answer in required form.

Question 6 (cont'd.)

6(a) (cont'd.)

(ii) Hence, find the co-ordinates of the point at which the slope of the tangent to the curve $y = f(x)$ is parallel to the line $x + 3y - 1 = 0$. **(5C)**

$$\text{Slope, } m = f'(x) = \frac{1}{1-x^2}$$

Slope of tangent parallel to line $x + 3y - 1 = 0$

$$x + 3y - 1 = 0$$
$$\Rightarrow 3y = -x + 1$$
$$\Rightarrow y = -\frac{1}{3}x + \frac{1}{3}$$
$$y = mx + c$$
$$\Rightarrow m = -\frac{1}{3}$$
$$\Rightarrow \frac{1}{1-x^2} = -\frac{1}{3}$$
$$\Rightarrow 1 - x^2 = -3$$
$$\Rightarrow -x^2 = -3 - 1 = -4$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$
$$\Rightarrow x = 2 \quad \text{as } x > 1$$

$$f(x) = \ln\sqrt{\frac{x+1}{x-1}}$$

@ $x = 2$

$$\Rightarrow f(2) = \ln\sqrt{\frac{2+1}{2-1}} = \ln\sqrt{\frac{3}{1}} = \ln\sqrt{3} \ \underline{\text{or}}\ \frac{1}{2}\ln 3$$

$$\Rightarrow \text{Co-ordinates} = (2, \ln\sqrt{3})$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for the equation of a line.
	–	Finds $m = -\frac{1}{3}$ <u>or</u> $f'(x) = -\frac{1}{3}$, but fails to progress.
High partial credit: (4 marks)		Correctly equates $f'(x) = \frac{1}{1-x^2} = -\frac{1}{3}$ and finds correct value(s) of x co-ordinate [accept ±2], but fails to find <u>or</u> finds incorrect value of y co-ordinate.

Question 6 (cont'd.)

6(b) Find the co-ordinates of the point of inflection of the curve $y = \frac{xe^{x+1}}{e^{2-x}}$. **(10D)**

$$y = \frac{xe^{x+1}}{e^{2-x}} = xe^{x+1-(2-x)} = xe^{x+1-2+x} = xe^{2x-1}$$

$$\Rightarrow \frac{dy}{dx} = x\frac{d}{dx}(e^{2x-1}) + (e^{2x-1})\frac{d}{dx}(x) = x(e^{2x-1})(2) + (e^{2x-1})(1) = (2x+1)e^{2x-1}$$

$$\frac{d^2y}{dx^2} = (2x+1)\frac{d}{dx}(e^{2x-1}) + (e^{2x-1})\frac{d}{dx}(2x+1) = (2x+1)(e^{2x-1})(2) + (e^{2x-1})(2) = 2(2x+1+1)e^{2x-1} = 2(2x+2)e^{2x-1}$$

@ point of inflection

$$\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 2(2x+2)e^{2x-1} = 0$$

$$\Rightarrow 2x+2 = 0$$

$$\Rightarrow 2x = -2$$

$$\Rightarrow x = -1$$

$$y = \frac{xe^{x+1}}{e^{2-x}}$$

@ $x = -1$

$$y = \frac{(-1)e^{-1+1}}{e^{2-(-1)}} = \frac{-e^0}{e^{2+1}} = -\frac{1}{e^3}$$

$$\Rightarrow \text{Co-ordinates} = \left(-1, -\frac{1}{e^3}\right)$$

Scale 10D (0, 4, 6, 8, 10)

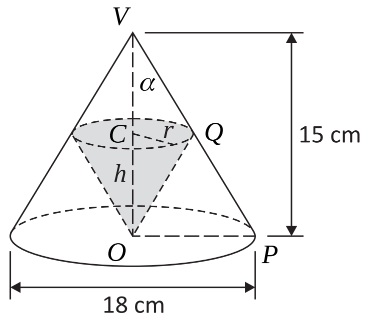
Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $\frac{d^2y}{dx^2} = 0$ at point of inflection <u>and stops</u>.
	–	Finds correctly $y = xe^{2x-1}$.
	–	Differentiates one term correctly.
Mid partial credit: (6 marks)	–	Finds $\frac{dy}{dx}$ correctly (simplified <u>or</u> not), but fails to progress.
	–	Finds $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ correctly, but not equated to zero.
High partial credit: (8 marks)	–	Finds both $\frac{dy}{dx}$ <u>and</u> $\frac{d^2y}{dx^2}$ correctly and equates to zero, but fails to finish fully, *e.g.* finds correct value of *x* co-ordinate, but fails to find <u>or</u> finds incorrect value of *y* co-ordinate.

Section B — Contexts and Applications — 150 marks

Answer **all three** questions from this section.

Question 7 **(45 marks)**

7(a) The diagram shows a right circular cone of radius 9 cm and height 15 cm. A smaller inverted cone of height h and radius r is inscribed within the larger cone.



(i) Using similar triangles, or otherwise, show that

$$h = \frac{45 - 5r}{3}. \qquad \textbf{(5C)}$$

$$\text{Diameter of cone} = 18 \text{ cm}$$
$$\Rightarrow \text{Radius of cone} = 9 \text{ cm}$$

ΔVOP and ΔVCQ are equiangular / similar ... as both Δs have common angle α, 90° angles and hence, the third angles in both Δs are equal

$$\Rightarrow \frac{9}{15} = \frac{r}{15 - h}$$
$$\Rightarrow 9(15 - h) = 15r$$
$$\Rightarrow 3(15 - h) = 5r$$
$$\Rightarrow 45 - 3h = 5r$$
$$\Rightarrow 3h = 45 - 5r$$
$$\Rightarrow h = \frac{45 - 5r}{3}$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* identifies one pair of corresponding sides or writes down $\tan\alpha = \frac{9}{15}$.
	–	Explains why triangles are similar.
High partial credit: (4 marks)	–	Finds $\frac{9}{15} = \frac{r}{15 - h}$, but fails to finish or finish incorrectly.

(ii) Express the volume of the smaller cone, in terms of π and r, in its simplest form. **(5C*)**

$$V_{\text{small}} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi r^2\left(\frac{45 - 5r}{3}\right)$$
$$= \frac{\pi r^2(45 - 5r)}{9} \text{ cm}^3 \ \underline{\text{or}} \ \frac{45\pi r^2 - 5\pi r^3}{9} \text{ cm}^3$$

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the volume of a cone.
High partial credit: (4 marks)	–	Substitutes fully correctly into volume formula, *i.e.* $V_{\text{small}} = \frac{1}{3}\pi r^2\left(\frac{45 - 5r}{3}\right)$, but fails to finish or finishes incorrectly.

* Deduct 1 mark off correct answer only for the omission of or incorrect use of units ('cm^3') - apply only once to each section (a), (b), (c), *etc*. of question.

Question 7 (cont'd.)

7(a) (cont'd.)

(iii) Find the maximum volume of the smaller cone, in terms of π. **(10D*)**

$$V_{small} = \frac{45\pi r^2 - 5\pi r^3}{9} \quad \text{... answer from part (a)(ii)}$$

$$\frac{dV}{dr} = \frac{d}{dr}\left(\frac{45\pi r^2 - 5\pi r^3}{9}\right)$$

$$= \frac{90\pi r - 15\pi r^2}{9}$$

Maximum volume when $\frac{dV}{dr} = 0$

$$\Rightarrow \frac{90\pi r - 15\pi r^2}{9} = 0$$

$$\Rightarrow 90\pi r - 15\pi r^2 = 0$$

$$\Rightarrow 15\pi r^2 = 90\pi r$$

$$\Rightarrow r = \frac{90}{15}$$

$$= 6 \text{ cm}$$

$$V_{small} = \frac{45\pi r^2 - 5\pi r^3}{9}$$

$$\Rightarrow V_{small\ (max)} = \frac{45\pi(6)^2 - 5\pi(6)^3}{9}$$

$$= \frac{1{,}620\pi - 1{,}080\pi}{9}$$

$$= \frac{540\pi}{9}$$

$$= 60\pi \text{ cm}^3$$

** Accept students' answers from part (a)(ii) for V_{small} if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down 'Maximum volume when $\frac{dV}{dr} = 0$'.
	–	Differentiates one term correctly.
Mid partial credit: (6 marks)		Differentiates correctly to find $\frac{dV}{dr}$, but fails to progress.
High partial credit: (8 marks)	–	Finds correct value of r, but fails to find or finds incorrect value for $V_{small\ (max)}$.
	–	Finds correct $V_{small\ (max)}$, but not in required form.

* Deduct 1 mark off correct answer only for the omission of or incorrect use of units ('cm^3') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 7 (cont'd.)

7(a) (cont'd.)

(iv) What fraction of the larger cone is unoccupied? **(5C)**

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

$$V_{\text{large}} = \frac{1}{3}\pi(9)^2(15) = \frac{1{,}215\pi}{3} = 405\pi$$

$$V_{\text{small (max)}} = 60\pi \qquad \text{... answer from part (a)(iii)}$$

$$\Rightarrow \text{Volume unoccupied} = 405\pi - 60\pi = 345\pi \text{ cm}^3$$

$\Rightarrow$ Fraction of the larger cone unoccupied

$$= \frac{345\pi}{405\pi} = \frac{69}{81} = \frac{23}{27}$$

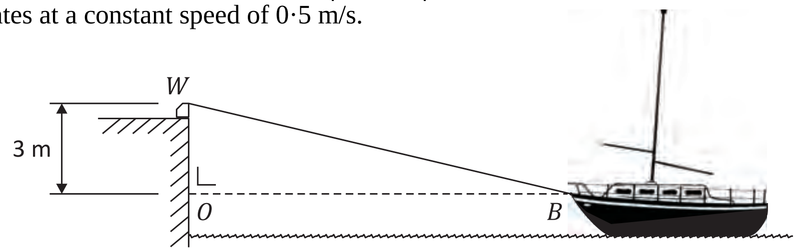
** Accept students' answers from part (a)(iii) for $V_{\text{small (max)}}$ if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the volume of a cone with some correct substitution.
	–	Finds correct value for V_{large}.
High partial credit: (4 marks)	–	Finds correct value of volume unoccupied, [ans. 345π], but fails to finish or finishes incorrectly.
	–	Finds fraction of larger cone occupied [ans. $\frac{4}{27}$], but fails to finish or finishes incorrectly, *i.e.* $1 - \frac{4}{27} = \frac{23}{27}$.

Question 7 (cont'd.)

7(b) A motorised winch is used to pull a boat into its berth position. The winch cable is attached to the bow (B) of the boat, as shown. The winch (W) is located on the quay 3 m above the bow of the boat and $|\angle WOB|$ is 90°.
The winch operates at a constant speed of 0·5 m/s.



(i) Let l be the length of the winch cable, $|WB|$.
Find x, the distance of the boat from the quay wall, in terms of l. **(5B)**

<u>Using Pythagoras' theorem</u>

$$|\text{Hyp}|^2 = |\text{Opp}|^2 + |\text{Adj}|^2$$
$$|WB| = l$$
$$|OB| = x$$
$$|WO| = 3$$
$$\Rightarrow \quad l^2 = x^2 + (3)^2$$
$$\Rightarrow \quad x^2 = l^2 - 9$$
$$\Rightarrow \quad x = \sqrt{l^2 - 9} \quad \underline{\text{or}} \quad (l^2 - 9)^{\frac{1}{2}} \quad \underline{\text{or}} \quad (l^2 - 9)^{0\cdot5}$$

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Substitutes correctly into Pythagoras' theorem, *i.e.* $l^2 = x^2 + (3)^2$, but fails to isolate <u>or</u> isolates x incorrectly.

(ii) Find the rate of change of x with respect to l. **(5B)**

$$\Rightarrow \quad x = \sqrt{l^2 - 9} \qquad \text{... answer from part (b)(i)}$$
$$\frac{dx}{dl} = \frac{d}{dl}(l^2 - 9)^{\frac{1}{2}}$$
$$= \frac{1}{2}(l^2 - 9)^{-\frac{1}{2}}(2l)$$
$$= \frac{l}{\sqrt{l^2 - 9}}$$

** Accept students' answers from part (b)(i) for x if not oversimplified.

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Some correct relevant differentiation, but incomplete, *e.g.* $\frac{dx}{dl} = \frac{1}{2}(l^2 - 9)^{-\frac{1}{2}}$, $\frac{1}{2}(l^2 - 9)^{\frac{1}{2}}(2l)$ <u>or</u> $(l^2 - 9)^{-\frac{1}{2}}(2l)$.

Question 7 (cont'd.)

7(b) (cont'd.)

(iii) Hence, find the speed at which the boat is approaching the quay wall when the length of the winch cable is 13 m. **(10D*)**

$$\frac{dl}{dt} = 0{\cdot}5 \text{ m/s}$$

$$x = \sqrt{l^2 - 9}$$

$$\Rightarrow \frac{dx}{dl} = \frac{l}{\sqrt{l^2 - 9}}$$ ... answer from part (b)(ii)

$$\frac{dx}{dt} = \frac{dx}{dl} \times \frac{dl}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{l}{\sqrt{l^2 - 9}} \times 0{\cdot}5$$

$$= \frac{l}{2\sqrt{l^2 - 9}}$$

@ $l = 13$

$$\Rightarrow \frac{dx}{dt} = \frac{13}{2\sqrt{(13)^2 - 9}}$$

$$= \frac{13}{2\sqrt{169 - 9}}$$

$$= \frac{13}{2\sqrt{160}}$$

$$= \frac{13\sqrt{10}}{80}$$

$$= 0{\cdot}513870...$$

$\cong$ 0·51 m/s <u>or</u> 0·514 m/s <u>or</u> 0·5139 m/s

** Accept students' answers from part (b)(ii) for $\frac{dx}{dl}$ if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $\frac{dl}{dt} = 0{\cdot}5$ <u>or</u> $\frac{dx}{dt} = \frac{dx}{dl} \times \frac{dl}{dt}$ <u>or similar</u>.
	–	Mentions a relevant rate of change, *i.e.* $\frac{dx}{dt}$ <u>and/or</u> $\frac{dx}{dl}$ <u>and/or</u> $\frac{dl}{dt}$.
Mid partial credit: (6 marks)	–	Finds $\frac{dx}{dt} = \frac{l}{\sqrt{l^2 - 9}} \times 0{\cdot}5$ <u>or</u> $\frac{l}{2\sqrt{l^2 - 9}}$, but fails to progress.
High partial credit: (8 marks)	–	Finds $\frac{dx}{dt} = \frac{13}{2\sqrt{(13)^2 - 9}}$, but fails to evaluate <u>or</u> evaluates incorrectly.

* Deduct 1 mark off correct answer only ❶ if final answer is not rounded <u>or</u> incorrectly rounded <u>or</u> ❷ for the omission of <u>or</u> incorrect use of units ('m/s') - apply only once to each section (a), (b), (c), *etc*. of question.

Question 8 **(55 marks)**

8(a) In the 100-metre race, sprinters typically reach their top speed about halfway through the race and try to maintain that speed for as long as possible.

A student analysed a sprinter's performance over the course of a particular race and determined that the speed of the sprinter can be approximated by the following model:

$$v(t) = \begin{cases} 0, & 0 \le t < 0{\cdot}15 \\ -0{\cdot}6t^2 + 5{\cdot}4t - k, & 0{\cdot}15 \le t < 4{\cdot}5 \\ 11{\cdot}3535, & t \ge 4{\cdot}5 \end{cases}$$

where v is the speed in metres per second, t is the time in seconds from the starting signal and k is a constant.

(i) Find the value of k. **(5C)**

Consider $0{\cdot}15 \le t < 4{\cdot}5$

$v(t) = -0{\cdot}6t^2 + 5{\cdot}4t - k$

@ $t = 4{\cdot}5$

$v(4{\cdot}5) = -0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - k$

$= 11{\cdot}3535$

$\Rightarrow -0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - k = 11{\cdot}3535$

$\Rightarrow k = -0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - 11{\cdot}3535$

$= -12{\cdot}15 + 24{\cdot}3 - 11{\cdot}3535$

$= 0{\cdot}7965$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down $v(4{\cdot}5) = 11{\cdot}3535$ <u>and stops</u>.
	–	Substitutes correctly into $v(t)$, *i.e.* $v(4{\cdot}5) = -0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - k$, but fails to equate to $11{\cdot}3535$.
High partial credit: (4 marks)	–	Equates correctly $v(4{\cdot}5) = 11{\cdot}3535$, *i.e.* $-0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - k = 11{\cdot}3535$, but fails to find <u>or</u> finds incorrect value of k.

(ii) Sketch the graph of v as a function of t for the first 7 seconds of the race. **(10D)**

❶ <u>Points:</u>

$v(t) = -0{\cdot}6t^2 + 5{\cdot}4t - 0{\cdot}7965$ ... answer from part (a)(i)

@ $t = 0{\cdot}15$

$v(0{\cdot}15) = 0$

@ $t = 1$

$\Rightarrow v(1) = -0{\cdot}6(1)^2 + 5{\cdot}4(1) - 0{\cdot}7965$

$= -0{\cdot}6 + 5{\cdot}4 - 0{\cdot}7965$

$= 4{\cdot}0035$

@ $t = 2$

$\Rightarrow v(2) = -0{\cdot}6(2)^2 + 5{\cdot}4(2) - 0{\cdot}7965$

$= -2{\cdot}4 + 10{\cdot}8 - 0{\cdot}7965$

$= 7{\cdot}6035$

@ $t = 3$

$\Rightarrow v(3) = -0{\cdot}6(3)^2 + 5{\cdot}4(3) - 0{\cdot}7965$

$= -5{\cdot}4 + 16{\cdot}2 - 0{\cdot}7965$

$= 10{\cdot}0035$

Question 8 (cont'd.)

8(a) (ii) (cont'd.)

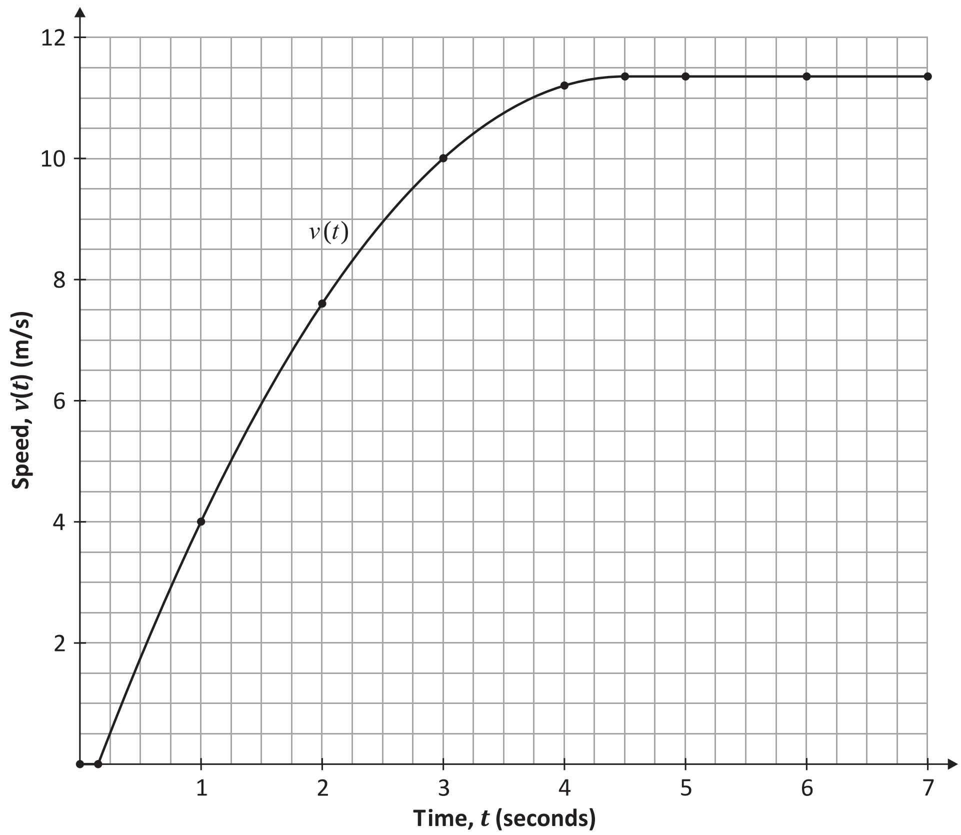
@ $t = 4$

$\Rightarrow$ $v(4) = -0{\cdot}6(4)^2 + 5{\cdot}4(4) - 0{\cdot}7965$
$= -9{\cdot}6 + 21.6 - 0{\cdot}7965$
$= 11{\cdot}2035$

@ $t = 4{\cdot}5$

$\Rightarrow$ $v(4{\cdot}5) = -0{\cdot}6(4{\cdot}5)^2 + 5{\cdot}4(4{\cdot}5) - 0{\cdot}7965$
$= -12{\cdot}15 + 24{\cdot}3 - 0{\cdot}7965$
$= 11{\cdot}3535$

❷ Points:



** Accept students' answers from part (a)(i) for $v(t)$ if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* evaluates $v(t)$ for any value between 0·15 and 4·5.
	–	Shows correct graph of $v(t)$ for $t = 4{\cdot}5$ to $t = 7$ (straight line) only.
Mid partial credit: (6 marks)	–	Evaluates $v(t)$ for several values between 0·15 and 4·5, but fails to plot graph <u>or</u> plots incorrect graph.
High partial credit: (8 marks)	–	Graph almost fully correct, but with one error/omission, *e.g.* graph begins rising from (0, 0) or straight lines(s) used instead of a curve.

Question 8 (cont'd.)

8(a) (cont'd.)

(iii) Find the distance travelled by the sprinter in the first 4·5 seconds of the race. **(10D*)**

Distance travelled in the interval [0, 4·5]

$$s(t) = 0 + \int_{0\cdot15}^{4\cdot5} v(t)\,dt$$

$$v(t) = -0\cdot6t^2 + 5\cdot4t - 0\cdot7965 \quad \text{... answer from part (a)(i)}$$

$$\Rightarrow s(t) = \int_{0\cdot15}^{4\cdot5} (-0\cdot6t^2 + 5\cdot4t - 0\cdot7965)\,dt$$

$$= -0\cdot6\frac{t^3}{3} + 5\cdot4\frac{t^2}{2} - 0\cdot7965t \Big|_{0\cdot15}^{4\cdot5}$$

$$= -0\cdot2t^3 + 2\cdot7t^2 - 0\cdot7965t \Big|_{0\cdot15}^{4\cdot5}$$

$$= [-0\cdot2(4\cdot5)^3 + 2\cdot7(4\cdot5)^2 - 0\cdot7965(4\cdot5)] - [-0\cdot2(0\cdot15)^3 + 2\cdot7(0\cdot15)^2 - 0\cdot7965(0\cdot15)]$$

$$= [-18\cdot225 + 54\cdot675 - 3\cdot58425] - [-0\cdot000675 + 0\cdot06075 - 0\cdot119475]$$

$$= [32\cdot86575] - [-0\cdot0594]$$

$$= 32\cdot92515 \text{ m}$$

** Accept students' answers from part (a)(i) for $v(t)$ if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down relevant integration formula for distance *i.e.* $s(t) = \int v(t)\,dt$ <u>and stops</u>.
	–	Some correct integration <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Integrates $v(t)$ correctly to find $s(t)$, *i.e.* $s(t) = -0\cdot6\frac{t^3}{3} + 5\cdot4\frac{t^2}{2} - 0\cdot7965t$, but no limits <u>or</u> incorrect limits used.
High partial credit: (8 marks)	–	Integrates $v(t)$ correctly with correct limits, *i.e.* $s(t) = -0\cdot6\frac{t^3}{3} + 5\cdot4\frac{t^2}{2} - 0\cdot7965t \Big|_{0\cdot15}^{4\cdot5}$, but fails to evaluate <u>or</u> evaluates incorrectly.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m') - apply only once to each section (a), (b), (c), *etc*. of question.

8(a) (cont'd.)

(iv) Hence, find the sprinter's finishing time for the race. Give your answer correct to three decimal places. **(5C*)**

Distance travelled in the interval [0, 4·5]

$= 32{\cdot}92515$ m ... answer from part (a)(iii)

Distance travelled in the interval [4·5, end of race]

$= 100 - 32{\cdot}92515$

$= 67{\cdot}07485$ m

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

$$\Rightarrow \text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$v(t \geq 4{\cdot}5) = 11{\cdot}3535$$

$$\Rightarrow t(t \geq 4{\cdot}5) = \frac{67{\cdot}07485}{11{\cdot}3535} = 5{\cdot}907856...$$

$$\Rightarrow \text{Total time} = 4{\cdot}5 + 5{\cdot}907856... = 10{\cdot}407856... \cong 10{\cdot}408 \text{ s}$$

** Accept students' answers from part (a)(iii) for distance travelled in the interval [0, 4·5] if not oversimplified.

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down relevant formula for speed with some correct substitution.
	–	Finds distance travelled after 4·5 seconds [ans. 100 – 32·92515 or answer from part (iii)].
High partial credit: (4 marks)	–	Finds correct value of $t(t \geq 4{\cdot}5)$ [ans. 5·908, 5·907856..., or $\frac{67{\cdot}07485}{11{\cdot}3535}$], but fails to finish or finishes incorrectly.

* Deduct 1 mark off correct answer only ❶ if final answer is not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('s') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 8 (cont'd.)

8(b) A model for an Olympic-standard 100 m sprinter was developed by mathematicians.
The speed of the sprinter may be calculated using the function:

$$w(t) = 11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t})$$

where t is the time in seconds from the starting signal.

(i) Find the maximum speed of the sprinter, correct to two decimal places. **(10D*)**

Maximum speed when $\frac{dw}{dt} = 0$

$$\begin{aligned}
& w(t) &=& \ 11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t}) \\
\Rightarrow\ & \frac{dw}{dt} &=& \ \frac{d}{dt}[11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t})] \\
& &=& \ 11{\cdot}7[0 - (e^{-0{\cdot}8t})(0{\cdot}8)] + 0{\cdot}03[0 - (e^{0{\cdot}3t})(0{\cdot}3)] \\
& &=& \ 11{\cdot}7(0{\cdot}8)e^{-0{\cdot}8t} - 0{\cdot}03(0{\cdot}3)e^{0{\cdot}3t} \\
& &=& \ 9{\cdot}36e^{-0{\cdot}8t} - 0{\cdot}009e^{0{\cdot}3t} \\
& &=& \ 0 \\
\Rightarrow\ & 9{\cdot}36e^{-0{\cdot}8t} - 0{\cdot}009e^{0{\cdot}3t} &=& \ 0 \\
\Rightarrow\ & 9{\cdot}36e^{-0{\cdot}8t} &=& \ 0{\cdot}009e^{0{\cdot}3t} \\
\Rightarrow\ & \frac{e^{0{\cdot}3t}}{e^{-0{\cdot}8t}} &=& \ \frac{9{\cdot}36}{0{\cdot}009} \\
\Rightarrow\ & e^{0{\cdot}3t + 0{\cdot}8t} &=& \ 1{,}040 \\
\Rightarrow\ & e^{1{\cdot}1t} &=& \ 1{,}040 \\
\Rightarrow\ & \text{In}(e^{1{\cdot}1t}) &=& \ \text{In}\,1{,}040 \\
\Rightarrow\ & 1{\cdot}1t &=& \ \text{In}\,1{,}040 \\
\Rightarrow\ & t &=& \ \frac{\ln 1{,}040}{1{\cdot}1} \\
& &=& \ 6{\cdot}315432...\text{ s}
\end{aligned}$$

Maximum speed when $t = 6{\cdot}315432...$

$$\begin{aligned}
& w(t) &=& \ 11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t}) \\
\Rightarrow\ & w(6{\cdot}315432...) &=& \ 11{\cdot}7(1 - e^{-0{\cdot}8(6{\cdot}315432...)}) + 0{\cdot}03(1 - e^{0{\cdot}3(6{\cdot}315432...)}) \\
& &=& \ 11{\cdot}7(0{\cdot}993605...) + 0{\cdot}03(-5{\cdot}650086...) \\
& &=& \ 11{\cdot}625186... - 0{\cdot}169502... \\
& &=& \ 11{\cdot}455683... \\
& &\cong& \ 11{\cdot}46\text{ m/s}
\end{aligned}$$

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down 'Maximum speed when $\frac{dw}{dt} = 0$'.
	–	Differentiates one term correctly, *e.g.* $11{\cdot}7[0 - (e^{-0{\cdot}8t})(0{\cdot}8)]$.
Mid partial credit: (6 marks)		Differentiates correctly to find $\frac{dw}{dt}$ and equates $\frac{dw}{dt} = 0$, but fails to isolate or isolates t incorrectly.
High partial credit: (8 marks)	–	Finds correct value of t [ans. 6·315432...], but fails to find or finds incorrect value for maximum speed.

* Deduct 1 mark off correct answer only ❶ if final answer is not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('m/s') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 8 (cont'd.)

8(b) (cont'd.)

(ii) Find an expression for the distance travelled by the sprinter after time t. **(10C)**

$$s(t) = \int w(t)\,dt$$

$$w(t) = 11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t})$$

$$\Rightarrow s(t) = \int [11{\cdot}7(1 - e^{-0{\cdot}8t}) + 0{\cdot}03(1 - e^{0{\cdot}3t})]\,dt$$

$$= 11{\cdot}7\left(t - \frac{e^{-0{\cdot}8t}}{-0{\cdot}8}\right) + 0{\cdot}03\left(t - \frac{e^{0{\cdot}3t}}{0{\cdot}3}\right) + c$$

$$= 11{\cdot}7t + 14{\cdot}625e^{-0{\cdot}8t} + 0{\cdot}03t - 0{\cdot}1e^{0{\cdot}3t} + c$$

$$= 11{\cdot}73t + 14{\cdot}625e^{-0{\cdot}8t} - 0{\cdot}1e^{0{\cdot}3t} + c$$

@ $t = 0$, $s = 0$

$$\Rightarrow s(0) = 11{\cdot}73(0) + 14{\cdot}625e^{-0{\cdot}8(0)} - 0{\cdot}1e^{0{\cdot}3(0)} + c$$

$$= 14{\cdot}625e^{0} - 0{\cdot}1e^{0} + c$$

$$= 14{\cdot}625(1) - 0{\cdot}1(1) + c$$

$$= 14{\cdot}525 + c$$

$$= 0$$

$$\Rightarrow 14{\cdot}525 + c = 0$$

$$\Rightarrow c = -14{\cdot}525$$

$$\Rightarrow s(t) = 11{\cdot}73t + 14{\cdot}625e^{-0{\cdot}8t} - 0{\cdot}1e^{0{\cdot}3t} - 14{\cdot}525$$

Scale 10C (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down relevant integration formula for distance *i.e.* $s(t) = \int w(t)\,dt$ <u>and stops</u>.
	–	Some correct integration <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (7 marks)	–	Integrates $w(t)$ correctly to find $s(t)$, *i.e.* $s(t) = 11{\cdot}73t + 14{\cdot}625e^{-0{\cdot}8t} - 0{\cdot}1e^{0{\cdot}3t}$, but fails to find <u>or</u> finds incorrect value of c.

Question 8 (cont'd.)

8(b) (cont'd.)

(iii) Hence, show that the sprinter completes the race in less than 10 seconds. **(5D)**

$$s(t) = 11{\cdot}73t + 14{\cdot}625e^{-0{\cdot}8t} - 0{\cdot}1e^{0{\cdot}3t} - 14{\cdot}525$$

@ $t = 10$

$$\Rightarrow s(10) = 11{\cdot}73(10) + 14{\cdot}625e^{-0{\cdot}8(10)} - 0{\cdot}1e^{0{\cdot}3(10)} - 14{\cdot}525$$
$$= 117{\cdot}3 + 14{\cdot}625e^{-8} - 0{\cdot}1e^{3} - 14{\cdot}525$$
$$= 117{\cdot}3 + 0{\cdot}004906... - 2{\cdot}008553... - 14{\cdot}525$$
$$= 100{\cdot}771352...$$

as $100{\cdot}771352... > 100$

$t_{\text{race}} < 10$ s

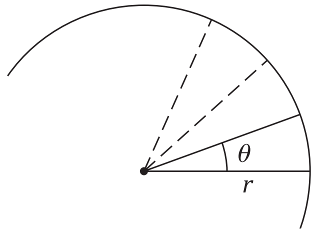
** Accept students' answers from part (b)(ii) for $s(t)$ if not oversimplified.

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down condition required, 'if $s(10) > 100$, then $t_{\text{race}} < 10$'.
	–	Some correct substitution into $s(10)$, <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (3 marks)	–	Fully correct substitution into $s(10)$, but fails to evaluate <u>or</u> evaluates incorrectly.
High partial credit: (4 marks)	–	Finds correct value for $s(10)$, but no conclusion <u>or</u> incorrect conclusion given.

Question 9 **(50 marks)**

9(a) A circular disc is divided into 12 unequal sectors whose areas are in arithmetic sequence. The area of the largest sector is twice that of the smallest sector. The radius of the disc is r and the acute angle in the smallest sector is θ, in degrees, as shown. The increase in angle in subsequent sectors is λ.



(i) Find the areas of the smallest and the largest sectors, in terms of r and θ. **(5B)**

Area of smallest sector $= \pi r^2\left(\frac{\theta}{360}\right)$

$= \frac{\pi r^2\theta}{360}$

$\Rightarrow$ Area of largest sector $= 2\left(\frac{\pi r^2\theta}{360}\right)$ <u>or</u> $\pi r^2\left(\frac{2\theta}{360}\right)$

$= \frac{\pi r^2\theta}{180}$

Scale 5B (0, 2, 5)

Partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the area of a sector.
	–	Finds correct area of one sector only.

(ii) Find an expression for the acute angle of the nth sector in the arithmetic sequence and hence, write down the size of the angle in the largest sector in terms of θ and λ. **(5C)**

❶ <u>Acute angle of the nth sector</u>

$T_n = a + (n-1)d$

$a = \theta$

$d = \lambda$

$\Rightarrow T_n = \theta + (n-1)\lambda$

❷ <u>Size of the angle in the largest sector</u>

$\Rightarrow T_n = \theta + (n-1)\lambda$

$T_{12} = \theta + (12-1)\lambda$

$= \theta + 11\lambda$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for T_n with some correct substitution (a <u>or</u> d).
	–	Correctly identifies $a = \theta$ and $d = \lambda$ with some correct substitution into formula for T_n (not stated).
	–	Finds expression for T_n by inspection <u>or</u> calculation <u>and stops</u>.
High partial credit: (4 marks)	–	Finds correct expression for T_n, but fails to find <u>or</u> finds incorrect expression for T_{12}.

Question 9 (cont'd.)

9(a) (cont'd.)

(iii) Find an equation for the sum of the acute angles in all of the sectors, in terms of θ and λ. **(5C)**

❶ <u>S_n, the sum of the first n angles</u>

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$n = 12$$
$$a = \theta$$
$$d = \lambda$$

$$\Rightarrow S_{12} = \frac{12}{2}[2\theta + (12-1)\lambda]$$
$$= 6[2\theta + 11\lambda]$$
$$= 12\theta + 66\lambda$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for S_n with some correct substitution (a <u>or</u> d).
	–	Correctly identifies $a = \theta$ and $d = \lambda$ with some correct substitution into formula for S_n (not stated).
High partial credit: (4 marks)	–	Finds correct expression for S_n, [ans. $\frac{n}{2}[2\theta + (n-1)\lambda]$], but fails to evaluate <u>or</u> incorrectly evaluates S_{12}.
	–	Finds correct expression for S_{12}, [ans. $\frac{12}{2}[2\theta + (12-1)\lambda]$], but fails to finish <u>or</u> finishes incorrectly.

Question 9 (cont'd.)

9(a) (cont'd.)

(iv) Use your answers to **parts (ii)** and **(iii)** above to find, in degrees, the value of θ. **(5C)**

❶

$T_{12} = \theta + 11\lambda$ ... answer from part (a)(ii)
$= 2\theta$ ... answer from part (a)(i)

$\Rightarrow \theta + 11\lambda = 2\theta$
$\Rightarrow 11\lambda = 2\theta - \theta$
$= \theta$
$\Rightarrow \lambda = \dfrac{\theta}{11}$

$\Rightarrow S_{12} = 12\theta + 66\lambda$
$= 360°$ ... answer from part (a)(iii)

$\Rightarrow 12\theta + 66\lambda = 360°$

Substituting $\lambda = \dfrac{\theta}{11}$ into equation:

$\Rightarrow 12\theta + 66\left(\dfrac{\theta}{11}\right) = 360°$
$\Rightarrow 12\theta + 6\theta = 360°$
$\Rightarrow 18\theta = 360°$
$\Rightarrow \theta = 20°$

<u>or</u>

❷

$T_{12} = \theta + 11\lambda$ ... answer from part (a)(ii)
$= 2\theta$ ... answer from part (a)(i)
$\Rightarrow \theta + 11\lambda = 2\theta$ ... ①

$\Rightarrow S_{12} = 12\theta + 66\lambda$
$= 360°$ ... answer from part (a)(iii)
$\Rightarrow 12\theta + 66\lambda = 360°$ ... ②

① $\theta + 11\lambda = 2\theta$ (×–6)
② $12\theta + 66\lambda = 360°$ (×1)

Equating ① and ②:

① $-6\theta - 66\lambda = -12\theta$
② $12\theta + 66\lambda = 360°$
$\Rightarrow 6\theta = 360° - 12\theta$
$\Rightarrow 18\theta = 360°$
$\Rightarrow \theta = 20°$

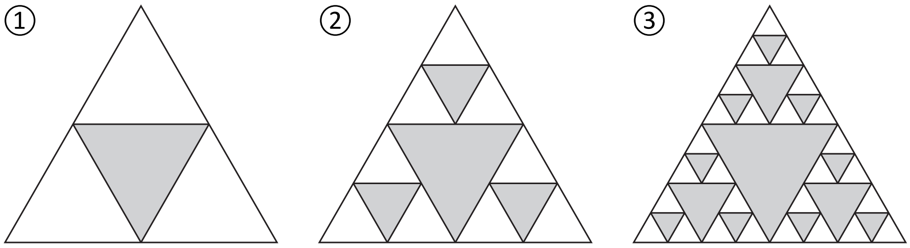
** Accept students' answers from previous parts if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down one expression in terms of θ and λ, *i.e.* $\theta + 11\lambda = 2\theta$ <u>or</u> $12\theta + 66\lambda = 360°$.
	–	Finds λ in terms of θ, *i.e.* $\frac{\theta}{11}$ [Method ❶], <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (4 marks)	–	Finds λ in terms of θ, *i.e.* $\frac{\theta}{11}$, and finds second expression, *i.e.* $12\theta + 66\lambda = 360°$ [Method ❶], but fails to finish <u>or</u> finishes incorrectly.
	–	Finds two expressions in terms of θ and λ, with work towards finding θ [Method ❷], but fails to finish <u>or</u> finishes incorrectly.

Question 9 (cont'd.)

9(b) An equilateral triangle can be subdivided into four smaller equilateral triangles of equal area. The first three patterns in a sequence of patterns are shown below. In each successive pattern, the unshaded triangle is subdivided into smaller equal triangles.



(i) Complete the table below to show the number of shaded and unshaded equilateral triangles in each pattern. **(5C)**

Pattern	**1**	**2**	**3**	**4**	**5**
Number of shaded triangles	1	4	**13**	**40**	**121**
Number of unshaded triangles	3	**9**	**27**	**81**	**243**

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	One, two or three correct entries.
High partial credit: (4 marks)	–	Four, five or six correct entries.

(ii) Write an expression in n for the number of unshaded triangles in the nth pattern in the sequence. **(5C)**

Number of unshaded triangles	3	**9**	**27**	**81**	**243**

Unshaded triangles: 3, 9, 27, 81, 243, ...

$\Rightarrow$ Geometric sequence

$T_n = ar^{n-1}$

$a = 3$

$r = 3$

$\Rightarrow T_n = 3(3^{n-1})$

$= 3^{1+n-1}$

$= 3^n$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for T_n with some correct substitution (a or r).
	–	Recognises terms in the sequence as 3 to the power of 1, 2, 3, *i.e.* 3^1, 3^2, 3^3, *etc*., but not the term in the nth pattern [3^n].
	–	Correctly identifies pattern as geometric sequence with correct a and r and some correct substitution into formula for T_n (not stated).
High partial credit: (4 marks)	–	Fully correct substitution into T_n, *i.e.* $T_n = 3(3^{n-1})$, but fails to finish or finishes incorrectly.

Question 9 (cont'd.)

9(b) (cont'd.)

(iii) Find an expression, in n, for the number of shaded triangles in the nth pattern in the sequence. **(10D)**

Number of shaded triangles	1	4	**13**	**40**	**121**

Change (1st difference) **3** **9** **27** **81**

$$T_1 = 1 = 3^0$$
$$T_2 = T_1 + 3^1 = 1 + 3^1$$
$$T_3 = T_2 + 3^2 = 1 + 3^1 + 3^2$$
$$T_4 = T_3 + 3^3 = 1 + 3^1 + 3^2 + 3^3$$
$$\Rightarrow T_n = 1 + 3^1 + 3^2 + 3^3 + \ldots + 3^{n-1} = 3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{n-1} = S_n(\text{Geometric series})\ 3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{n-1}$$
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$a = 1$$
$$r = 3$$
$$\Rightarrow S_n(3^0 + 3^1 + 3^2 + 3^3 + \ldots + 3^{n-1}) = \frac{1(1-3^n)}{1-3} = \frac{1-3^n}{-2} = \frac{3^n - 1}{2}$$
$$\Rightarrow T_n = \frac{3^n - 1}{2}$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $T_2 = 1 + 3^1$, $T_3 = 1 + 3^1 + 3^2$, *etc.* in the pattern.
	–	Identifies that 3^{n-1} added to previous term to get current term.
Mid partial credit: (6 marks)	–	Finds $T_n = \text{sum}(1 + 3^1 + 3^2 + \ldots + 3^{n-1})$, but fails to progress.
High partial credit: (8 marks)	–	Finds $T_n = \text{sum}(1 + 3^1 + 3^2 + \ldots + 3^{n-1})$ with some correct substitution (a or r) into S_n formula, but fails to finish or finishes incorrectly.

9(b) (cont'd.)

(iv) Find the fraction of the overall area that is shaded in the 5th pattern. **(5C)**

Area of 1st pattern $= \frac{1}{4}(1)$

Area of 2nd pattern $= \frac{1}{4} + \frac{1}{4}(\frac{1}{4})(3)$

$= \frac{1}{4} + \frac{3}{16}$

Area of 3rd pattern $= \frac{1}{4} + \frac{1}{4}(\frac{1}{4})(3) + \frac{1}{4}(\frac{1}{4})(\frac{1}{4})(9)$

$= \frac{1}{4} + \frac{3}{16} + \frac{9}{64}$

Pattern: $=$ ① $\frac{1}{4}$, ② $\frac{1}{4} + \frac{3}{16}$, ③ $\frac{1}{4} + \frac{3}{16} + \frac{9}{64}$, ...

❶ Continuing pattern:

$\Rightarrow$ Area of 4th pattern $= \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{9(3)}{64(4)}$

$\Rightarrow$ Area of 5th pattern $= \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \frac{27(3)}{256(4)}$

$= \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \frac{81}{1{,}024}$

$= \frac{256 + 3(64) + 9(16) + 27(4) + 81}{1{,}024}$

$= \frac{781}{1{,}024}$

or

❷ Sum of geometric progression:

$S_n = \frac{a(1 - r^n)}{1 - r}$ $\quad n = 5,\ a = \frac{1}{4},\ r = \frac{3}{4}$

$\Rightarrow S_n = \frac{\frac{1}{4}\left(1 - \left(\frac{3}{4}\right)^5\right)}{1 - \frac{3}{4}}$

$= 1 - \left(\frac{3}{4}\right)^5$

$= 1 - \frac{243}{1{,}024}$

$= \frac{781}{1{,}024}$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* finds fraction of area shaded in first three patterns.
	–	Some correct substitution (a or r) into S_n formula and stops or fails to progress.
High partial credit: (4 marks)	–	Substantive work towards finding shaded area in 5th pattern, *e.g.* correct area for 4th pattern, but fails to finish or finishes incorrectly.
	–	Fully correct substitution (a and r) into S_n formula, but fails to finish or finishes incorrectly.

Question 9 (cont'd.)

9(b) (cont'd.)

(v) In which pattern will the shaded area be greater than 95% of the overall area? **(5C)**

Pattern: = ① $\frac{1}{4}$, ② $\frac{1}{4}+\frac{3}{16}$, ③ $\frac{1}{4}+\frac{3}{16}+\frac{9}{64}$, ...

Sum of geometric progression

$$S_n = \frac{a(1-r^n)}{1-r} \qquad a=\frac{1}{4},\ r=\frac{3}{4}$$

$$\Rightarrow S_n = \frac{\frac{1}{4}\left(1-\left(\frac{3}{4}\right)^n\right)}{1-\frac{3}{4}}$$

$$= 1-\left(\frac{3}{4}\right)^n$$

$$= 0{\cdot}95$$

$$\Rightarrow 1-\left(\frac{3}{4}\right)^n = 0{\cdot}95$$

$$\Rightarrow \left(\frac{3}{4}\right)^n = 1-0{\cdot}95$$

$$= 0{\cdot}05$$

$$\Rightarrow n = \log_{\frac{3}{4}}(0{\cdot}05)$$

$$= 10{\cdot}413343...$$

$$\Rightarrow n = 11$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down that pattern is the sum of a geometric progression and identifies correct a and r.
	–	Some correct substitution (a or r) into S_n formula and stops or fails to progress.
High partial credit: (4 marks)	–	Fully correct substitution (a and r) into S_n formula and finds $1-\left(\frac{3}{4}\right)^n = 0{\cdot}95$, but fails to finish or finishes incorrectly.
	–	Finds $n = 10{\cdot}413343...$, but fails to round up to $n = 11$.

Pre-Leaving Certificate Examination, 2018

Mathematics

Higher Level – Paper 2
Marking Scheme (300 marks)

Structure of the Marking Scheme

Students' responses are marked according to different scales, depending on the types of response anticipated.
Scales labelled A divide students' responses into two categories (correct and incorrect).
Scales labelled B divide responses into three categories (correct, partially correct, and incorrect), and so on.
These scales and the marks that they generate are summarised in the following table:

Scale label	**A**	**B**	**C**	**D**
No. of categories	2	3	4	5
5 mark scale		**0, 2, 5**	**0, 2, 4, 5**	**0, 2, 3, 4, 5**
10 mark scale			**0, 4, 7, 10**	**0, 4, 6, 8, 10**
15 mark scale				**0, 5, 9, 12, 15**

A general descriptor of each point on each scale is given below. More specific directions in relation to interpreting the scales in the context of each question are given in the scheme, where necessary.

Marking scales – level descriptors

A-scales (two categories)
- incorrect response (no credit)
- correct response (full credit)

B-scales (three categories)
- response of no substantial merit (no credit)
- partially correct response (partial credit)
- correct response (full credit)

C-scales (four categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

D-scales (five categories)
- response of no substantial merit (no credit)
- response with some merit (low partial credit)
- response about half-right (mid partial credit)
- almost correct response (high partial credit)
- correct response (full credit)

In certain cases, typically involving ❶ incorrect rounding, ❷ omission of units, ❸ a misreading that does not oversimplify the work <u>or</u> ❹ an arithmetical error that does not oversimplify the work, a mark that is one mark below the full-credit mark may also be awarded. Such cases are flagged with an asterisk.
Thus, for example, scale **10C*** indicates that 9 marks may be awarded.

- The * for units to be applied only if the student's answer is fully correct.
- The * to be applied once only **<u>within each section (a), (b), (c), *etc.*</u>** of all questions.
- The * penalty is not applied for the omission of units in currency solutions.

Unless otherwise specified, accept correct answer with <u>or</u> without work shown.

Accept students' work in one part of a question for use in subsequent parts of the question, unless this oversimplifies the work involved.

Summary of Marks – 2018 LC Maths (Higher Level, Paper 2)

Section A

Q.1	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
	(b)		10D (0, 4, 6, 8, 10)
			25

Q.2	**(a)**		10D (0, 4, 6, 8, 10)
	(b)		5C* (0, 2, 4, 5)
	(c)		10D* (0, 4, 6, 8, 10)
			25

Q.3	**(a)**		5C (0, 2, 4, 5)
	(b)		5C (0, 2, 4, 5)
	(c)		15D (0, 5, 9, 12, 15)
			25

Q.4	**(a)**		10D (0, 4, 6, 8, 10)
	(b)		5C* (0, 2, 4, 5)
	(c)		10D (0, 4, 6, 8, 10)
			25

Q.5	**(a)**	**(i)**	5C (0, 2, 4, 5)
		(ii)	10D (0, 4, 6, 8, 10)
	(b)		10D (0, 4, 6, 8, 10)
			25

Q.6	**(a)**	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
	(b)		10D* (0, 4, 6, 8, 10)
			25

Section B

Q.7	**(a)**	**(i)**	10C* (0, 4, 7, 10)
		(ii)	5B* (0, 2, 5)
	(b)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
	(c)	**(i)**	10D* (0, 4, 6, 8, 10)
		(ii)	5D (0, 2, 3, 4, 5)
			50

Q.8	**(a)**	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
	(b)	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	5C (0, 2, 4, 5)
		(iii)	5C (0, 2, 4, 5)
		(iv)	5C (0, 2, 4, 5)
		(v)	5D (0, 2, 3, 4, 5)
			50

Q.9	**(a)**	**(i)**	10D (0, 4, 6, 8, 10)
		(ii)	10D (0, 4, 6, 8, 10)
		(iii)	5D (0, 2, 3, 4, 5)
		(iv)	5C (0, 2, 4, 5)
		(v)	10D* (0, 4, 6, 8, 10)
	(b)		10D* (0, 4, 6, 8, 10)
			50

Current Marking Scheme

Assumptions about these marking schemes on the basis of past SEC marking schemes should be avoided. While the underlying assessment principles remain the same, the exact details of the marking of a particular type of question may vary from a similar question asked by the SEC in previous years in accordance with the contribution of that question to the overall examination in the current year. In setting these marking schemes, we have strived to determine how best to ensure the fair and accurate assessment of students' work and to ensure consistency in the standard of assessment from year to year. Therefore, aspects of the structure, detail and application of the marking schemes for these examinations are subject to change from past SEC marking schemes and from one year to the next without notice.

Pre-Leaving Certificate Examination, 2018

Mathematics

Higher Level – Paper 2
Marking Scheme (300 marks)

General Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	3 questions

Answer **all nine** questions.

Marks will be lost if all necessary work is not clearly shown.

Answers should include the appropriate units of measurement, where relevant.

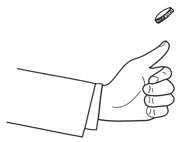
Answers should be given in simplest form, where relevant.

Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

1(a) Orla and Liam play a game that consists of tossing an unbiased coin. The first person to get a 'heads' is the winner. If Orla tosses first, find the probability that:



(i) Liam wins the game on his first toss, **(5C)**

P(Liam wins on 1st toss)

$$= P(\text{Orla loses on 1st toss}) + P(\text{Liam wins on 1st toss})$$
$$= \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{4} \ \underline{\text{or}} \ 0{\cdot}25$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Liam wins, *e.g.* 'P(Orla loses on 1st toss) + P(Liam wins on 1st toss) <u>or</u> P(L, W)' <u>or similar</u> <u>and stops</u>.
	–	Correct probabilities chosen, but incorrect operator used.
High partial credit: (4 marks)	–	Correct probabilities and operator chosen, *i.e.* P(Liam wins) = $\frac{1}{2} \times \frac{1}{2}$, but fails to express as a single fraction <u>or equivalent</u>.

1(a) (cont'd.)

(ii) Orla wins the game on her second toss, **(5C)**

P(Orla wins on 2nd toss)

$= P(\text{Orla loses on 1st toss}) + P(\text{Liam loses on 1st toss}) + P(\text{Orla wins on 2nd toss})$

$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{8}$ <u>or</u> $0{\cdot}125$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Orla wins, *e.g.* 'P(Orla loses on 1st toss) + P(Liam loses on 1st toss) + P(Orla wins on 2nd toss)' <u>or similar</u> <u>and stops</u>.
	–	Correct probabilities chosen, but incorrect operator used.
High partial credit: (4 marks)	–	Correct probabilities and operator chosen, *i.e.* P(Orla wins) = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$, but fails to express as a single fraction <u>or equivalent</u>.

(iii) Orla wins the game. **(5C)**

P(Orla wins)

$= P(\text{Orla wins on 1st toss}) + P(\text{Orla wins on 2nd toss}) + P(\text{Orla wins on 3rd toss}) + ...$

$= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + ...$

$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) + ...$

$\Rightarrow$ Infinite geometric progression

$S_\infty = \frac{a}{1-r}$ $\qquad a = \frac{1}{2},\ r = \frac{1}{4}$

$\Rightarrow$ P(Orla wins) $= \dfrac{\frac{1}{2}}{1-\frac{1}{4}}$

$= \frac{4}{6}$ <u>or</u> $\frac{2}{3}$ $0{\cdot}666666...$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability that Orla wins, *e.g.* 'P(Orla wins on 1st toss) + P(Orla wins on 2nd toss) + P(Orla wins on 3rd toss) + ...' <u>or similar</u>.
	–	Finds P(Orla wins) = $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + ...$
High partial credit: (4 marks)	–	Recognises that P(Orla wins) is equal to the sum of a G.P. with $a = \frac{1}{2}$ and $r = \frac{1}{4}$, but fails to find or finds incorrect S_∞.

Question 1 (cont'd.)

1(b) A game of chance comprises a player spinning a 'lottery wheel'. There are 100 positions in which the ball has an equal chance of landing but there is only one chance for a player to win the top prize.
Find the minimum number of spins which Carina must attempt in order that the probability of winning the top prize at least once is no less than 25%. **(10D)**

$P(\text{wins top prize}) = \dfrac{1}{100}$

$\Rightarrow P(\text{does not win}) = 1 - \dfrac{1}{100}$

$= \dfrac{99}{100}$

$\Rightarrow P(\text{Carina wins top prize at least once in } n \text{ attempts})$

$= 1 - P(\text{never wins top prize in } n \text{ attempts})$

$= 1 - \left(\dfrac{99}{100}\right)^n$

$= 0{\cdot}25$

$\Rightarrow 1 - \left(\dfrac{99}{100}\right)^n = 0{\cdot}25$

$\Rightarrow \left(\dfrac{99}{100}\right)^n = 1 - 0{\cdot}25$

$= 0{\cdot}75$

$\Rightarrow n = \log_{\frac{99}{100}}(0{\cdot}75)$

$= 28{\cdot}624125...$

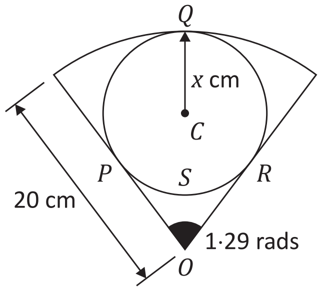
$\Rightarrow n = 29$

Scale 10D (0, 4, 6, 8, 10)

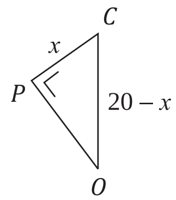
Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct explanation of probability, *e.g.* 'P(Carina wins at least once) = 1 – P(never wins top prize) <u>or similar</u> <u>and stops</u>.
	–	Finds P(does not win) = $1 - \dfrac{1}{100}$ <u>or</u> $\dfrac{99}{100}$.
Mid partial credit: (6 marks)	–	Finds P(Carina wins top prize) and finds $1 - \left(\dfrac{99}{100}\right)^n = 0{\cdot}25$ <u>or</u> $1 - \left(\dfrac{99}{100}\right)^n > 0{\cdot}25$ <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (8 marks)	–	Finds P(Carina wins top prize), *i.e.* $1 - \left(\dfrac{99}{100}\right)^n = 0{\cdot}25$ <u>or</u> $1 - \left(\dfrac{99}{100}\right)^n > 0{\cdot}25$ with substantive work towards finding n, but fails to finish <u>or</u> finishes incorrectly.
	–	Finds $n = 28{\cdot}624125...$, but fails to round up to $n = 29$.

Question 2 **(25 marks)**

The diagram below shows a sector of a circle with centre O and radius 20 cm. A circle with centre C and radius x cm lies within the sector and touches it at P, Q and R. S is another point on the circle.
$|\angle POR| = 1{\cdot}29$ radians.



2(a) By considering the triangle POC, show that x is equal to 7·5 cm, correct to one decimal place. **(10D)**



Consider ΔPOC

$|\angle POC| = \dfrac{1{\cdot}29}{2}$

$= 0{\cdot}645$ rads

Using trigonometry

$\sin|\angle POC| = \dfrac{|PC|}{|OC|}$

$|PC| = x$

$|OC| = 20 - x$

$\Rightarrow \sin 0{\cdot}645 = \dfrac{x}{20-x}$

$\Rightarrow 0{\cdot}601198... = \dfrac{x}{20-x}$

$\Rightarrow x = (20 - x)(0{\cdot}601198...)$

$= 12{\cdot}023968... - 0{\cdot}601198...x$

$\Rightarrow x + 0{\cdot}601198...x = 12{\cdot}023968...$

$\Rightarrow 1{\cdot}601198...x = 12{\cdot}023968...$

$\Rightarrow x = \dfrac{12{\cdot}023968...}{1{\cdot}601198...}$

$= 7{\cdot}509355...$

$\cong 7{\cdot}5$ cm

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down or indicates on diagram that ΔPOC is a right-angled triangle with $|\angle CPO| = 90°$.
	–	Finds $|\angle POC| = 0{\cdot}645$ rads.
	–	Finds $|OC| = 20 - x$.
	–	Some correct substitution into trig ratio (sin) and stops or fails to progress.
Mid partial credit: (6 marks)	–	Finds $\sin 0{\cdot}645 = \dfrac{x}{20-x}$ and stops or fails to progress.
High partial credit: (8 marks)	–	Finds $0{\cdot}601198... = \dfrac{x}{20-x}$ with some work towards finding x, but fails to finish or finishes incorrectly.

Question 2 (cont'd.)

2(b) Hence, find the area of the region which is inside the sector but outside the circle, correct to three decimal places. **(5C*)**

Area of sector	=	$\frac{1}{2}r^2\theta$	... θ in radians
	=	$\frac{1}{2}(20)^2(1\cdot29)$	
	=	258 cm^2	
Area of circle	=	πr^2	
	=	$\pi(7\cdot5)^2$	
	=	$176\cdot714586... \text{ cm}^2$	
$\Rightarrow$ Area outside circle	=	$258 - 176\cdot714586...$	
	=	$81\cdot285413... \text{ cm}^2$	
	$\cong$	$81\cdot285 \text{ cm}^2$	

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for the area of a sector or a circle with some correct substitution into formula.
	–	Finds Area of sector [ans. 258].
	–	Finds Area of circle [ans. 176·714586...].
High partial credit: (4 marks)	–	Finds area = $\frac{1}{2}(20)^2(1\cdot29) - \pi(7\cdot5)^2$, but fails to evaluate or evaluates incorrectly.
	–	Finds both areas separately [$\frac{1}{2}(20)^2(1\cdot29)$ and $\pi(7\cdot5)^2$] with one error/omission, but finishes correctly.

* Deduct 1 mark off correct answer only ❶ if final answer is not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('cm^2') - apply only once to each section (a), (b), (c), *etc*. of question.

Question 2 (cont'd.)

2(c) Find the perimeter of the region *PORS* bounded by the arc *PSR* and the lines *OP* and *OR*. Give your answer correct to the nearest cm. **(10D*)**

Perimeter of the region *PORS*

$= |PO| + |OR| + |\text{arc } RSP|$

Consider ΔPOC

Using Pythagoras' theorem

$|\text{Hyp}|^2 = |\text{Opp}|^2 + |\text{Adj}|^2$

$\Rightarrow |OC|^2 = |CP|^2 + |PO|^2$

$|OC| = 20 - 7{\cdot}5$

$= 12{\cdot}5$

$|CP| = 7{\cdot}5$

$\Rightarrow |PO|^2 = (12{\cdot}5)^2 - (7{\cdot}5)^2$

$= 156{\cdot}25 - 56{\cdot}25$

$= 100$

$\Rightarrow |PO| = \sqrt{100}$

$= 10$ cm

Also $|OR| = 10$ cm

$|\angle RCP| = 2|\angle OCP|$

$|\angle OCP| = \pi - \frac{\pi}{2} - 0{\cdot}645$

$= \frac{\pi}{2} - 0{\cdot}645$

$\Rightarrow |\angle RCP| = 2(\frac{\pi}{2} - 0{\cdot}645)$

$= \pi - 1{\cdot}29$

$= 1{\cdot}851592...$ (rads)

$|\text{arc } RSP| = r\theta$

$= (7{\cdot}5)(1{\cdot}851592...)$

$= 13{\cdot}886944...$ cm

Perimeter of the region *PORS*

$= 10 + 10 + 13{\cdot}886944...$

$= 33{\cdot}886944...$

$\cong 34$ cm

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for Pythagoras' theorem or for the length of an arc (in rads) with some correct substitution into formula.
	–	Indicates Perimeter of the region *PORS* $= |PO| + |OR| + |\text{arc } RSP|$ and stops.
	–	Correct substitution into formula for Pythagoras' theorem (not stated), but fails to evaluate or evaluates incorrectly.
	–	Writes down that $|\angle RCP| = 2|\angle OCP|$ or equivalent.
Mid partial credit: (6 marks)	–	Finds correct value of $|PO|$ or $|OR|$ only.
	–	Finds correct value of $|\text{arc } RSP|$ only.
High partial credit: (8 marks)	–	Finds correct value of $|PO|$ (or $|OR|$) or $|\text{arc } RSP|$ with substantive work towards finding the value of other length, but fails to finish or finishes incorrectly.
	–	Finds correct value of $|PO|$ (or $|OR|$) and $|\text{arc } RSP|$, but fails to find perimeter or finds incorrect value of perimeter.

* Deduct 1 mark off correct answer only if final answer is not rounded or incorrectly rounded - apply only once to each section (a), (b), (c), *etc.* of question.

Question 3 **(25 marks)**

Two circles, k_1 and k_2, touch externally.

3(a) The equation of the circle k_1 is $x^2 + y^2 - 6x + 2y - 15 = 0$.
Find the centre and radius of k_1. **(5C)**

❶ Centre of k_1:

General equation of a circle:
$s: x^2 + y^2 + 2gx + 2fy + c = 0$ with centre $(-g, -f)$

k_1: $x^2 + y^2 - 6x + 2y - 15 = 0$
$x^2 + y^2 + 2(-3)x + 2(1)y - 15 = 0$

$\Rightarrow$ Centre $(-g, -f) = (3, -1)$

❷ Radius of k_1:

$$r_1(\text{radius of } k_1) = \sqrt{g^2 + f^2 - c}$$
$$= \sqrt{(-3)^2 + (1)^2 - (-15)}$$
$$= \sqrt{9 + 1 + 15}$$
$$= \sqrt{25}$$
$$= 5$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* effort at relating one or more coefficients of given equation to general equation of a circle.
	–	Effort at completing square(s).
High partial credit: (4 marks)	–	Finds either centre <u>or</u> radius correctly.
	–	Substantive work towards finding centre <u>and</u> radius, but with one critical error, *e.g.* centre (–3, –1) and finds radius with incorrect value.

Question 3 (cont'd.)

3(b) The centres of the two circles lie on the line $4x + 3y - 9 = 0$. The radius of circle k_2 is 10 units. If the co-ordinates of the centre of circle k_2 are expressed in the form $(-g, -f)$, show that $(3 + g)^2 + (f - 1)^2 = 225$. **(5C)**

Let c_1 be the centre of k_1 and c_2 be the centre of k_2

r_2(radius of k_2) = 10
c_2(centre of k_2) = $(-g, -f)$
r_1(radius of k_1) = 5 ... answer from part (a)
c_1(centre of k_1) = $(3, -1)$ ... answer from part (a)

$$r_1 + r_2 = |c_1c_2|$$
$$\Rightarrow 5 + 10 = \sqrt{(3-(-g))^2 + (-1-(-f))^2}$$
$$\Rightarrow 15 = \sqrt{(3+g)^2 + (-1+f)^2}$$
$$= \sqrt{(3+g)^2 + (f-1)^2}$$
$$\Rightarrow (3+g)^2 + (f-1)^2 = 15^2$$
$$= 225$$

** Accept students' answers for r_1 and c_1 from part (a) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down that $r_1 + r_2 = |c_1c_2|$ or similar.
	–	Finds correct value of $r_1 + r_2$ [ans. 15].
	–	Some correct substitution into distance formula to find $|c_1c_2|$ and stops or fails to progress.
High partial credit: (4 marks)	–	Substitutes correctly into $r_1 + r_2 = |c_1c_2|$, but fails to find correct expression.

Question 3 (cont'd.)

3(c) Hence, or otherwise, find the possible equations of k_2. **(15D)**

$$(-g, -f) \in 4x + 3y - 9 = 0$$
$$\Rightarrow 4(-g) + 3(-f) - 9 = 0$$
$$\Rightarrow 3f = -4g - 9$$
$$\Rightarrow f = \frac{-4g - 9}{3} \quad \text{... ①}$$
$$225 = (3 + g)^2 + (f - 1)^2 \quad \text{... ②} \quad \text{... from part (b)}$$

Substituting ① into ②:

$$\Rightarrow 225 = (3 + g)^2 + \left(\frac{-4g - 9}{3} - 1\right)^2$$
$$= g^2 + 6g + 9 + \left(\frac{-4g - 12}{3}\right)^2$$
$$= g^2 + 6g + 9 + \frac{16g^2 + 96g + 144}{9}$$
$$\Rightarrow 225(9) = (g^2 + 6g + 9)(9) + 16g^2 + 96g + 144$$
$$\Rightarrow 2{,}025 = 9g^2 + 54g + 81 + 16g^2 + 96g + 144$$
$$\Rightarrow 25g^2 + 150g - 1{,}800 = 0$$
$$\Rightarrow g^2 + 6g - 72 = 0$$
$$\Rightarrow (g - 6)(g + 12) = 0$$

$$\Rightarrow g - 6 = 0 \qquad \Rightarrow g + 12 = 0$$
$$\Rightarrow g = 6 \qquad \Rightarrow g = -12$$
$$\Rightarrow f = \frac{-4g - 9}{3} = \frac{-4(6) - 9}{3} = \frac{-33}{3} = -11 \qquad \Rightarrow f = \frac{-4g - 9}{3} = \frac{-4(-12) - 9}{3} = \frac{39}{3} = 13$$
$$\Rightarrow c_2: (-6, 11) \qquad \Rightarrow c_2: (12, -13)$$
$$\Rightarrow k_2: (x + 6)^2 + (y - 11)^2 = 100 \qquad \Rightarrow k_2: (x - 12)^2 + (y + 13)^2 = 100$$

Scale 15D (0, 5, 9, 12, 15)

Low partial credit: (5 marks)	–	Any relevant first step, *e.g.* substitutes $(-g, -f)$ correctly into $4x + 3y - 9 = 0$.
	–	Finds $f = \frac{-4g - 9}{3}$ <u>or</u> $g = \frac{-3f - 9}{4}$ <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (9 marks)	–	Substitutes f <u>or</u> g correctly into equation $225 = (3 + g)^2 + (f - 1)^2$, but fails to find correct quadratic equation.
High partial credit: (12 marks)	–	Find two correct values of first variable, but fails to find corresponding values of second variable.
	–	Find only one correct value of first variable and corresponding value of second variable.
	–	Finds both variables, but fails to find <u>or</u> finds incorrect equations of circles.

Question 4 **(25 marks)**

4(a) Find the equation of the line l through the point (–3, 2), which divides the line segment (–6, 2) to (–3, –4) internally in the ratio 1:2. **(10D)**

Line segment (–6, 2) to (–3, –4) divided internally in the ratio 1:2
(x_1, y_1) (x_2, y_2) $k_1 : k_2$

$$P(x, y) = \left(\frac{k_1x_2 + k_2x_1}{k_1 + k_2}, \frac{k_1y_2 + k_2y_1}{k_1 + k_2}\right)$$
$$= \left(\frac{1(-3) + 2(-6)}{1 + 2}, \frac{1(-4) + 2(2)}{1 + 2}\right)$$
$$= \left(\frac{-15}{3}, \frac{0}{3}\right)$$
$$= (-5, 0)$$

Line l contains (–3, 2) and (–5, 0)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\Rightarrow \quad m_l \text{ (slope of line } l) = \frac{0 - 2}{-5 - (-3)}$$
$$= \frac{-2}{-2}$$
$$= 1$$

Equation of l
(–3, 2), $m_l = 1$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - (2) = 1(x - (-3))$$
$$\Rightarrow \quad y - 2 = x + 3$$
$$\Rightarrow \quad x - y + 5 = 0$$

Equation of l
(–5, 0), $m_l = 1$

$$y - y_1 = m(x - x_1)$$
$$\Rightarrow \quad y - (0) = 1(x - (-5))$$
$$\Rightarrow \quad y = x + 5$$
$$\Rightarrow \quad x - y + 5 = 0$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for ratio with some correct substitution into formula.
	–	Identifies correct relevant formula for slope <u>or</u> equation of a line with some correct substitution of (–3, 2) into formula.
Mid partial credit: (6 marks)	–	Substitutes correctly into ratio formula, <u>and stops</u> <u>or</u> fails to progress.
	–	Finds one ordinate only (correct).
	–	Correct co-ordinates, but no work shown.
High partial credit: (8 marks)	–	Finds correct slope of line l, but fails to find equation of line l <u>or</u> finds incorrect equation of line l.
	–	Finds equation of line l with one error / omission, but finishes correctly.

Question 4 (cont'd.)

4(b) Find the co-ordinates of the points where l cuts the x-axis and the y-axis and hence, find the area of the triangle formed by l and the two axes. **(5C*)**

❶ Co-ordinates of points where l cuts the x-axis and y-axis

l: $x - y + 5 = 0$

x-axis

$\Rightarrow y = 0$
$\Rightarrow x - 0 + 5 = 0$
$\Rightarrow x = -5$
$\Rightarrow$ cuts the x-axis at $(-5, 0)$

y-axis

$\Rightarrow x = 0$
$\Rightarrow 0 - y + 5 = 0$
$\Rightarrow -y = -5$
$\Rightarrow y = 5$
$\Rightarrow$ cuts the y-axis at $(0, 5)$

❷ Area of triangle formed by l and the two axes

$$\Rightarrow \text{Area of } \Delta = \frac{1}{2}|x_1y_2 - x_2y_1|$$
$$= \frac{1}{2}|(-5)(5) - (0)(0)|$$
$$= \frac{1}{2}|-25|$$
$$= \frac{1}{2}(25)$$
$$= 12{\cdot}5 \text{ units}^2$$

Scale 5C* (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for the area of a triangle with some correct substitution.
	–	Finds either correct x or y intercept and stops or fails to progress.
High partial credit: (4 marks)	–	Finds both x and y intercepts correctly, but fails to find or finds incorrect area of triangle.
	–	Finds area of triangle with one incorrect intercept/error/omission, but finishes correctly.

* Deduct 1 mark off correct answer only for the omission of or incorrect use of units ('units²') - apply only once to each section (a), (b), (c), *etc*. of question.

Question 4 (cont'd.)

4(c) A second line, $y = mx + c$, where m and c are positive constants, passes through (–3, 2) and forms a triangle with the axes of equal area to that in part **(b)** above.
Find the equation of this line. **(10D)**

$y = mx + c$

❶ $(-3, 2) \in y = mx + c$

$\Rightarrow -3m + c = 2$

$\Rightarrow c = 3m + 2$ ... ①

❷ Intercepts x-axis @ $y = 0$

$y = mx + c$

$\Rightarrow 0 = mx + c$

$\Rightarrow mx = -c$

$\Rightarrow x = \dfrac{-c}{m}$

$\Rightarrow$ cuts the x-axis at $(\dfrac{-c}{m}, 0)$

Intercepts y-axis @ $x = 0$

$y = m(0) + c$

$\Rightarrow y = c$

$\Rightarrow$ cuts the y-axis at $(0, c)$

❸ Area of $\Delta = \dfrac{1}{2}|x_1y_2 - x_2y_1|$

$= \dfrac{1}{2}|(\dfrac{-c}{m})(c) - (0)(0)|$

$= \dfrac{c^2}{2m}$

$= 12{\cdot}5$ ... answer from part (b)

$\Rightarrow \dfrac{c^2}{2m} = 12{\cdot}5$

$\Rightarrow c^2 = 12{\cdot}5(2m)$

$= 25m$ ... ②

❹ Substituting ① into ②:

$(3m + 2)^2 = 25m$

$\Rightarrow 9m^2 + 12m + 4 = 25m$

$\Rightarrow 9m^2 - 13m + 4 = 0$

$\Rightarrow (9m - 4)(m - 1) = 0$

$\Rightarrow 9m - 4 = 0$ $\Rightarrow m - 1 = 0$

$\Rightarrow m = \dfrac{4}{9}$ $\Rightarrow m = 1$

$\Rightarrow$ slope of line l

$c = 3m + 2$ ... ①

$\Rightarrow c = 3(\dfrac{4}{9}) + 2$

$= \dfrac{10}{3}$

❺ Equation of line

$y = mx + c$

$\Rightarrow y = \dfrac{4}{9}x + \dfrac{10}{3}$ or $4x - 9y + 30 = 0$

Question 4 (cont'd.)

4(c) (cont'd.)

** Accept students' answers for Area of Δ from part (b) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* substitutes (–3, 2) correctly into $y = mx + c$. Finds either correct x <u>or</u> y intercept <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Substitutes correctly into area of triangle formula and finds $\frac{c^2}{2m} = 12{\cdot}5$ <u>or</u> $c^2 = 25m$, but fails to find correct quadratic equation.
High partial credit: (8 marks)	– –	Finds correct slope, *i.e.* $m = \frac{4}{9}$, but fails to find equation of line <u>or</u> finds incorrect equation of line. Finds equation of line with one error/omission, but finishes correctly.

Question 5 **(25 marks)**

5(a) A jury of 12 people is to be selected from a panel of 8 men and 8 women.

(i) In how many ways can the jury be selected? **(5C)**

Possible jurors $= 8 + 8$
$= 12$

$\#$ total juries $= \dbinom{16}{12}$
$= {}^{16}C_{12}$
$= \dfrac{16!}{12!\,(16-12)!}$
$= \dfrac{16 \times 15 \times 14 \times 13}{4 \times 3 \times 2 \times 1}$
$= \dfrac{43{,}680}{24}$
$= 1{,}820$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down '$\#$ total juries $= \dbinom{16}{12}$ or ${}^{16}C_{12}$' and stops or fails to progress.
	–	Writes down or evaluates correctly ${}^{16}P_{12}$ [ans. $16 \times 15 \times 14 \times 13$ or 43,680].
High partial credit: (4 marks)	–	Finds $\dfrac{16!}{12!\,(16-12)!}$ or $\dfrac{16 \times 15 \times 14 \times 13}{4 \times 3 \times 2 \times 1}$, but fails to evaluate or evaluates incorrectly.

Question 5 (cont'd.)

5(a) (cont'd.)

(ii) Find the probability that the jury selected has more women than men. **(10D)**

More women than men

$\Rightarrow$ Jury composition = (8 women + 4 men) or (7 women + 5 men)

$$\#\text{juries} = \left[\binom{8}{8}\times\binom{8}{4}\right]+\left[\binom{8}{7}\times\binom{8}{5}\right]$$
$$= \left[{}^8C_8 \times {}^8C_4\right]+\left[{}^8C_7 \times {}^8C_5\right]$$
$$= \left[1\times\frac{8\times7\times6\times5}{4\times3\times2\times1}\right]+\left[8\times\frac{8\times7\times6}{3\times2\times1}\right]$$
$$= [1 \times 70] + [8 \times 56]$$
$$= 70 + 448$$
$$= 518$$

$\#$ total juries = 1,820 ... answer from part (a)(i)

$$\Rightarrow P(\text{more women}) = \frac{518}{1{,}820}$$
$$= \frac{37}{130} \ \text{or} \ 0{\cdot}284615...$$

** Accept students' answers for # total juries from part (a)(i) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down $\#\text{juries} = \binom{8}{8}\times\binom{8}{4}$, ${}^8C_8 \times {}^8C_4$, $\binom{8}{7}\times\binom{8}{5}$ or ${}^8C_7 \times {}^8C_5$ (evaluated or not).
	–	Finds $\left[\binom{8}{8}\times\binom{8}{4}\right]\times\left[\binom{8}{7}\times\binom{8}{5}\right]$, but fails to evaluate or evaluates incorrectly.
	–	Writes downs or evaluates correctly $\left[{}^8P_8 \times {}^8P_4\right] \times \left[{}^8P_7 \times {}^8P_5\right]$.
Mid partial credit: (6 marks)	–	Finds $\left[\binom{8}{8}\times\binom{8}{4}\right]+\left[\binom{8}{7}\times\binom{8}{5}\right]$, but fails to evaluate or evaluates incorrectly.
	–	Finds $\left[\binom{8}{8}\times\binom{8}{4}\right]\times\left[\binom{8}{7}\times\binom{8}{5}\right]$ and evaluates correctly [ans. 31,360].
High partial credit: (8 marks)	–	Finds # juries = 518, but fails to find or finds incorrect probability.

Question 5 (cont'd.)

5(b) A Maths teacher tells her class of 23 students that "There is a greater than 50% chance of two or more of you having the same birthday."
Do you agree with her? Justify your answer by calculation. **(10D)**

P(2 or more have the same birthday)

$$= 1 - P(\text{none have the same birthday})$$
$$= 1 - \left[\left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\left(\frac{362}{365}\right)\ldots\left(\frac{344}{365}\right)\left(\frac{343}{365}\right)\right]$$
$$= 1 - \frac{(365)(364)(363)(362)\ \ldots\ (344)(343)}{365^{23}}$$
$$= 1 - 0{\cdot}492702\ldots$$
$$= 0{\cdot}507279\ldots$$

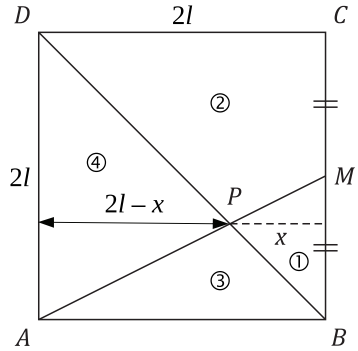
as $0{\cdot}507279\ldots > 50\%$,
the teacher is correct

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down 'P(2 or more have the same birthday) = 1 – P(none have the same birthday)' <u>or similar</u> <u>and stops</u>.
	–	Finds $\left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\ldots\left(\frac{343}{365}\right)$, but fails to evaluate <u>or</u> evaluates incorrectly.
Mid partial credit: (6 marks)	–	Finds P(2 or more have the same birthday) $=1 - \left[\left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\ldots\left(\frac{343}{365}\right)\right]$, but fails to evaluate <u>or</u> evaluates incorrectly.
	–	Finds $\left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\ldots\left(\frac{343}{365}\right)$, and evaluates correctly [ans. 0·492702...].
High partial credit: (8 marks)	–	Finds correct P(2 or more have same birthday) [ans. 0·507279...], but no conclusion <u>or</u> incorrect conclusion given.

Question 6 **(25 marks)**

6(a) The diagram shows a square $ABCD$ and the point M, the midpoint of $[BC]$. $[AM]$ and $[BD]$ intersect at the point P and divide the square into four regions. x is the perpendicular height of the triangle BMP.



(i) Let $|AB| = 2l$.
Find an equation for the sum of the areas of the four regions, in terms of x and l, and hence, show that $x = \frac{2l}{3}$. **(10D)**

❶

Area of region ① $= \text{Area of } \Delta BMP$
$= \frac{1}{2}(\text{base} \times \perp\text{height})$
$= \frac{1}{2}(l)(x)$
$= \frac{1}{2}lx$

Area of region ② $= \text{Area of } \Delta BCD - \text{Area of } \Delta BMP$
$= \frac{1}{2}(2l)(2l) - \frac{1}{2}lx$
$= 2l^2 - \frac{1}{2}lx$

Area of region ③ $= \text{Area of } \Delta ABM - \text{Area of } \Delta BMP$
$= \frac{1}{2}(2l)(l) - \frac{1}{2}lx$
$= l^2 - \frac{1}{2}lx$

Area of region ④ $= \text{Area of } \Delta APD$
$= \frac{1}{2}(2l)(2l - x)$
$= 2l^2 - lx$

Equation for the sum of the areas of the four regions
$= \text{Area of regions ① + ② + ③ + ④}$
$= \frac{1}{2}lx + 2l^2 - \frac{1}{2}lx + l^2 - \frac{1}{2}lx + 2l^2 - lx$
$= 5l^2 - \frac{3}{2}lx$

❷

Total area of square $= (2l)(2l)$
$= 4l^2$

$\Rightarrow 5l^2 - \frac{3}{2}lx = 4l^2$

$\Rightarrow \frac{3}{2}lx = 5l^2 - 4l^2$
$= l^2$

$\Rightarrow x = \frac{2}{3}l$

Question 6 (cont'd.)

6(a) (i) (cont'd.)

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	– –	Any relevant first step, *e.g.* finds correct area of one or two regions. Finds Total area of square [ans. $4l^2$].
Mid partial credit: (6 marks)	–	Finds correct equation for the sum of the areas of all four regions [ans. $5l^2 - \frac{3}{2}lx$] and stops or fails to progress.
High partial credit: (8 marks)	–	Equates the sum of the areas to the Total area of square, *i.e.* $5l^2 - \frac{3}{2}lx = 4l^2$, but fails to finish or finishes incorrectly.

(ii) Hence, find the ratio of the areas of the four regions. **(5C)**

❶ Area of region ① $= \frac{1}{2}lx$

$= \frac{1}{2}l(\frac{2}{3}l) = \frac{1}{3}l^2$

Area of region ② $= 2l^2 - \frac{1}{2}lx$

$= 2l^2 - \frac{1}{2}l(\frac{2}{3}l) = \frac{5}{3}l^2$

Area of region ③ $= l^2 - \frac{1}{2}lx$

$= l^2 - \frac{1}{2}l(\frac{2}{3}l) = \frac{2}{3}l^2$

Area of region ④ $= 2l^2 - lx$

$= 2l^2 - l(\frac{2}{3}l) = \frac{4}{3}l^2$

❷ Ratio of areas of region ① : region ② : region ③ : region ④

$= \frac{1}{3}l^2 : \frac{5}{3}l^2 : \frac{2}{3}l^2 : \frac{4}{3}l^2$

$= 1 : 5 : 2 : 4$

** Accept students' answers for the area of each region from part (a)(i) if not oversimplified.

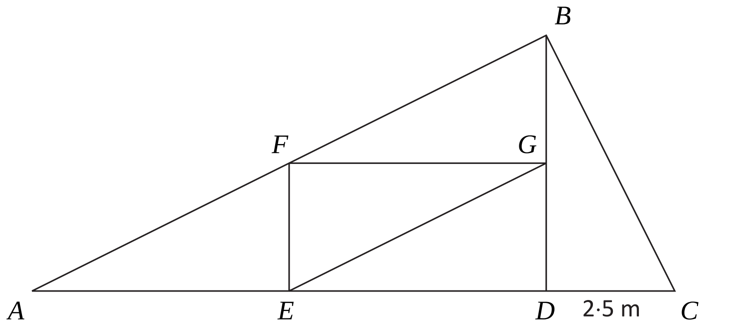
Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* finds correct area of one or two regions in terms of l^2.
High partial credit: (4 marks)	–	Finds correct area of all four regions in terms of l^2, but fails to find ratio or finds incorrect ratio.

Question 6 (cont'd.)

6(b) The diagram shows the support framework for the roof configuration of a new house. The design is based on a larger triangle which is subdivided into five identical triangles that are similar to the larger triangle.

Each of the frameworks, or *roof trusses*, is constructed using timber. The quantity surveyor for the project needs to determine the total length of timber required for each truss.



Given that the shortest side of each of the smaller triangles in the design is 2·5 m, find the total length of timber required to make each truss.
Give your answer in the form $a + b\sqrt{c}$, where a, b and $c \in \mathbb{N}$. **(10D*)**

$\Delta DCB \equiv \Delta DGE \equiv \Delta GBF \equiv \Delta FEG \equiv \Delta EFA$

$|DC| = 2{\cdot}5$ m
$|DG| = |DC| = 2{\cdot}5$ m
also $|GB| = 2{\cdot}5$ m
$\Rightarrow |DB| = 2{\cdot}5 + 2{\cdot}5 = 5$ m

Consider ΔBCD
<u>Using Pythagoras' theorem</u>

$$|\text{Hyp}|^2 = |\text{Opp}|^2 + |\text{Adj}|^2$$
$$\Rightarrow |BC|^2 = |DC|^2 + |DB|^2$$
$$\Rightarrow |BC|^2 = (2{\cdot}5)^2 + (5)^2 = 6{\cdot}25 + 25 = 31{\cdot}25 = \frac{125}{4}$$
$$\Rightarrow |BC| = \sqrt{\frac{125}{4}} = \frac{5\sqrt{5}}{2}\text{ m}$$

Total length of timber required

$$= 4(5) + 2(2{\cdot}5) + 4\left(\frac{5\sqrt{5}}{2}\right) = 20 + 5 + 10\sqrt{5} = 25 + 10\sqrt{5}\text{ m}$$

Scale 10D (0, 4, 6, 8, 10*)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down formula for Pythagoras' theorem with some correct substitution into formula.
	–	Finds correct value of $\|DB\|$ [ans. 5].
Mid partial credit: (6 marks)	–	Finds correct value of $\|BC\|$ [ans. $\frac{5\sqrt{5}}{2}$ <u>or</u> equivalent] <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (8 marks)	–	Finds correct value of $\|BC\|$ and almost correct total length, but includes one error/omission, *e.g.* one extra/missing side in answer.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('units^2') - apply only once to each section (a), (b), (c), *etc*. of question.

Section B	Contexts and Applications	150 marks

Answer **all three** questions from this section.

Question 7 **(50 marks)**

Mean sea level is the midpoint between high tide and low tide and it is used as a datum from which all altitudes are measured. On a particular day, mean sea level in a boating marina first occurs at midnight (i.e. $t = 0$). The expected depth of water in the marina, in metres, can be modelled using the trigonometric function:

$$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56,$$

where t is the time in hours from midnight and $\left(\frac{\pi}{6}t\right)$ is expressed in radians.

7(a) (i) Find the time at which the first high tide occurs and the depth of the water in the marina at that time. **(10C*)**

❶ <u>Time</u>

$$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56$$

$\Rightarrow$ High tide occurs when $\sin\left(\frac{\pi}{6}t\right) = 1$

$\Rightarrow \frac{\pi}{6}t = \sin^{-1}(1)$

$\Rightarrow \frac{\pi}{6}t = \frac{\pi}{2}$

$\Rightarrow t = \frac{6}{2}$

$= 3$ hours

$\Rightarrow$ Time $=$ 03:00 <u>or</u> 3:00 <u>or</u> 3:00 am

❷ <u>Depth of water</u>

$$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56$$

@ 03:00

$\sin\left(\frac{\pi}{6}t\right) = 1$

$\Rightarrow h_{\text{high tide}} = 1{\cdot}46(1) + 1{\cdot}56$

$= 1{\cdot}46 + 1{\cdot}56$

$\Rightarrow h_{\text{high tide}} = 3{\cdot}02$ m

Scale 10C* (0, 4, 7, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down that maximum / high tide occurs when $\sin A$ <u>or</u> $\sin\left(\frac{\pi}{6}t\right) = 1$.
	–	Attempts to differentiate $h(t)$.
	–	Finds $\sin\frac{\pi}{6}t = 1$ <u>or</u> $\frac{\pi}{6}t = \sin^{-1}(1)$, but fails to find t <u>or</u> finds incorrect t, *e.g.* error using radians.
High partial credit: (7 marks)	–	Finds one correct answer only (Time = 03:00 <u>or</u> $h_{\text{high tide}} = 3{\cdot}02$ m).

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 7 (cont'd.)

7(a) (cont'd.)

(ii) Find, by calculation, the period of $h(t)$. **(5B*)**

General equation of a sine function:

$$f(t) = a + b\sin ct$$

$$\text{Period} = \frac{2\pi}{c}$$

$$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56$$

$$\Rightarrow c = \frac{\pi}{6}$$

$$\Rightarrow \text{Period} = \frac{2\pi}{\frac{\pi}{6}}$$

$$= 2\pi\left(\frac{6}{\pi}\right)$$

$$= 12 \text{ hours}$$

Scale 5B* (0, 2, 5)

Partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for the period of a trig function <u>or</u> general equation of a sine function with notation.
	–	Some correct use of 2π <u>or</u> $\frac{\pi}{6}$, *e.g.* $2\pi \div x$ <u>or</u> $x \div \frac{\pi}{6}$, $x \neq 2\pi$ <u>or</u> $\frac{\pi}{6}$.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('hours') - apply only once in each section (a), (b), (c), *etc*. of question.

7(b) **(i)** Use the depth function, $h(t)$, to show the expected depth of water in the marina between midnight and the following midnight. **(10D)**

$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56$									
Time	0:00	3:00	6:00	9:00	12:00	15:00	18:00	21:00	00:00
***t* (hours)**	0	3	**6**	**9**	**12**	**15**	**18**	**21**	**24**
***h*(*t*) (m)**	<u>**1·56**</u>	<u>**3·02**</u>	<u>**1·56**</u>	<u>**0·10**</u>	<u>**1·56**</u>	<u>**3·02**</u>	<u>**1·56**</u>	<u>**0·10**</u>	<u>**1·56**</u>

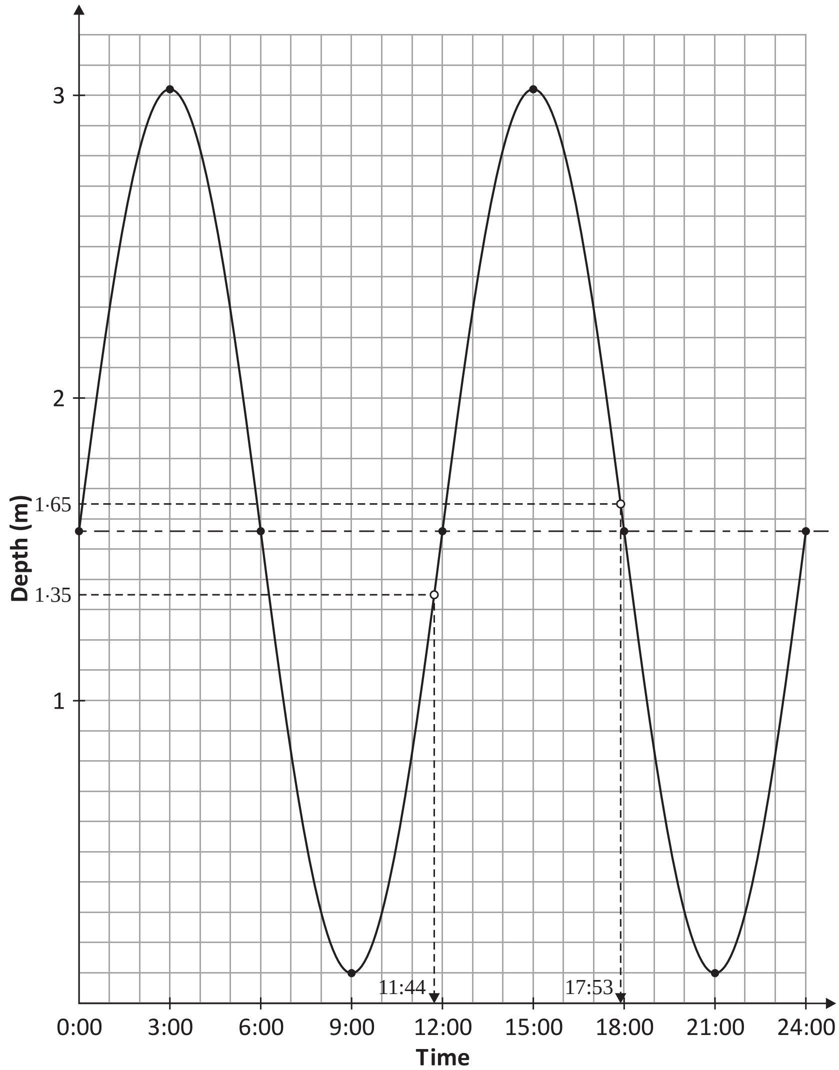
Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Finds one <u>or</u> two correct depths. [Accept incorrect answer from part (a)(i)].
Mid partial credit: (6 marks)	–	Finds three, four <u>or</u> five correct depths.
High partial credit: (8 marks)	–	Finds six, seven <u>or</u> eight correct depths.

Question 7 (cont'd.)

7(b) (cont'd.)

(ii) Sketch the graph of $h(t)$ between midnight and the following midnight. **(5C)**



Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Plots one <u>or</u> two correct points. [Accept incorrect answer from part (a)(i)].
High partial credit: (4 marks)	– –	Plots at least seven correct points. Plots all points, but graph not sketched <u>or</u> sketched incorrectly.

(iii) A large cruiser wishes to enter the marina to refuel. The boat requires a minimum water level of 1·35 m. When it is fully fuelled, the boat requires at least 1·65 m. Use your graph to estimate the time interval for which the cruiser can enter the marina in order that it is not grounded on the sea-bed if refuelling takes 4·5 hours. **(5C)**

<u>From graph:</u>
Latest departure time = 17:53 (±0:30)
Earliest entry time = 11:44 (±0:30)

$\Rightarrow$ Time interval = [11:44, 17:53 – 4:30]
= [11:44 (±0:30), 13:23 (±0:30)]

Question 7 (cont'd.)

7(b) (iii) (cont'd.)

** Accept answers based on students' graph in part (b)(ii) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	– –	Any relevant first step, *e.g.* indicates clearly depths of 1·35 m and/or 1·65 m on the graph with corresponding intercepts and times, but no values given. Identifies correct latest departure time or earliest entry time, *i.e.* 11:44 or 17:53.
High partial credit: (4 marks)	– – –	Identifies correct latest departure time and earliest entry time from graph, but fails to find or finds incorrect time interval. Finds correct answer for time interval, but no work shown on graph. Final answer outside of tolerance, but work shown on student's graph.

7(c) (i) Find the rate at which the depth of the water in the marina is changing at 8:00 a.m., correct to two decimal places. Explain your answer in the context of the question. **(10D*)**

❶ Rate at which the depth of the water is changing

$$h(t) = 1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56$$

$$h'(t) = \frac{d}{dt}(1{\cdot}46\sin\left(\frac{\pi}{6}t\right) + 1{\cdot}56)$$

$$= (1{\cdot}46)(\frac{\pi}{6})\cos\left(\frac{\pi}{6}t\right)$$

@ $t = 8$

$$\Rightarrow h'(8) = (1{\cdot}46)(\frac{\pi}{6})\cos\frac{8\pi}{6}$$

$$= \frac{73\pi}{300}\cos\frac{4\pi}{3}$$

$$= \frac{73\pi}{300}\left(-\frac{1}{2}\right)$$

$$= -\frac{73\pi}{600}$$

$$= -0{\cdot}382227...$$

$$\cong -0{\cdot}38 \text{ m/hr}$$

❷ Explanation

Answer – the tide is going out and the water level is dropping by 0·38 m per hour

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* some correct effort at differentiation.
Mid partial credit: (6 marks)	–	Finds $h'(t) = (1{\cdot}46)(\frac{\pi}{6})\cos\left(\frac{\pi}{6}t\right)$, but fails to evaluate or evaluates incorrectly.
High partial credit: (8 marks)	–	Finds $h'(8) = -0{\cdot}382227...$ or $-0{\cdot}38$, but fails to contextualise answer properly.

* Deduct 1 mark off correct answer only ❶ if final answer is not rounded or incorrectly rounded or ❷ for the omission of or incorrect use of units ('m/hr') - apply only once to each section (a), (b), (c), *etc.* of question.

Question 7 (cont'd.)

7(c) (cont'd.)

(ii) Hence, find the other times at which the depth of the water is changing at the same rate. **(5D)**

$h'(t) = (1\cdot46)(\frac{\pi}{6})\cos\left(\frac{\pi}{6}t\right)$ ... answer from part (c)(i)

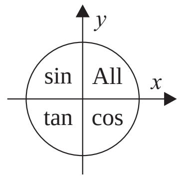
$= -\frac{73\pi}{600}$

$\Rightarrow (1\cdot46)(\frac{\pi}{6})\cos\left(\frac{\pi}{6}t\right) = -\frac{73\pi}{600}$

$\Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$

Reference angle, $\alpha = \cos^{-1}\frac{1}{2}$

$= \frac{\pi}{3}$



① $\Rightarrow \frac{\pi}{6}t = \pi - \frac{\pi}{3}$ ... 2nd quadrant

$= \frac{2\pi}{3}$

$\Rightarrow t = 4$ <u>or</u> 4:00 <u>or</u> 04:00 <u>or</u> 4:00 am

② $\Rightarrow \frac{\pi}{6}t = \pi + \frac{\pi}{3}$ ... 3rd quadrant

$= \frac{4\pi}{3}$

$\Rightarrow t = 8$ <u>or</u> 8:00 <u>or</u> 08:00 <u>or</u> 8:00 am

③ $\Rightarrow \frac{\pi}{6}t = \frac{2\pi}{3} + 2\pi$

$= \frac{8\pi}{3}$

$\Rightarrow t = 16$ <u>or</u> 16:00 <u>or</u> 4:00 pm

④ $\Rightarrow \frac{\pi}{6}t = \frac{4\pi}{3} + 2\pi$

$= \frac{10\pi}{3}$

$\Rightarrow t = 20$ <u>or</u> 20:00 <u>or</u> 8:00 pm

$\Rightarrow$ Other times = 04:00, 16:00, 20:00 (<u>or</u> equivalent)

** Accept students' answers for $h'(t)$ from part (c)(i) if not oversimplified.

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down 20:00 as answer (taken from graph).
	–	Finds $(1\cdot46)(\frac{\pi}{6})\cos\left(\frac{\pi}{6}t\right) = -\frac{73\pi}{600}$ <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (3 marks)		Finds correct reference angle.
High partial credit: (4 marks)	–	Finds two correct values of t (excluding 08:00).

Question 8 **(50 marks)**

8(a) A recent flood damaged a warehouse that contained 3000 computers. An insurance assessor, trying to estimate the damage, takes a random sample of 140 computers and finds that 34 of them are damaged.

(i) Create a 95% confidence interval for the proportion of computers that are damaged. **(10D)**

Confidence interval for a population proportion, p, is

$$= \left[\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

$x = 34$

$n = 140$

$\Rightarrow \quad \hat{p} =$ population proportion observed

$$= \frac{34}{140}$$

$$= \frac{17}{70} \text{ or } 0{\cdot}242857...$$

At 95% confidence interval

z-value $= 1{\cdot}96$

$\Rightarrow$ 95% confidence interval for this population proportion (p)

$$= \left[\frac{17}{70} - 1{\cdot}96\sqrt{\frac{\frac{17}{70}\left(1-\frac{17}{70}\right)}{140}},\ \frac{17}{70} + 1{\cdot}96\sqrt{\frac{\frac{17}{70}\left(1-\frac{17}{70}\right)}{140}}\right]$$

$$= \left[\frac{17}{70} - 1{\cdot}96\sqrt{\frac{\frac{17}{70}\left(\frac{53}{70}\right)}{140}},\ \frac{17}{70} + 1{\cdot}96\sqrt{\frac{\frac{17}{70}\left(\frac{53}{70}\right)}{140}}\right]$$

$= [0{\cdot}2428... - 0{\cdot}0710...,\ 0{\cdot}2428... + 0{\cdot}0710...]$

$= [0{\cdot}1718...,\ 0{\cdot}3138...]$

$\cong [0{\cdot}1718,\ 0{\cdot}3138]$

i.e. can be 95% confident that the proportion of computers damaged lies in the range $17{\cdot}18\% < p < 31{\cdot}38\%$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct formula for confidence interval, *i.e.* $\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ or $\hat{p} \pm 1{\cdot}96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$, and stops.
	–	Finds correct value for observed population proportion ($\hat{p}$) and stops or fails to progress.
	–	Mention of 5% level of significance and therefore comparing to z-value of $\pm 1{\cdot}96$.
Mid partial credit: (6 marks)	–	Finds correct value for $\hat{p}$ and some correct substitution into 95% confidence interval for population proportion.
High partial credit: (8 marks)	–	Correct substitution into 95% confidence interval, but fails to finish or finishes incorrectly.

Question 8 (cont'd.)

8(a) (cont'd.)

(ii) Find the 95% confidence interval for the number of computers that are damaged. **(5C)**

95% confidence interval for this population proportion, p, is

$= [17{\cdot}18, 31{\cdot}38]$ ... answer from part (a)(i)

$\Rightarrow$ 95% confidence interval for the number of computers damaged (x)

$$= \left[3{,}000\left(\frac{17{\cdot}18}{100}\right),\ 3{,}000\left(\frac{31{\cdot}38}{100}\right)\right]$$

$$= [515{\cdot}4,\ 941{\cdot}4]$$

$$\cong [516,\ 942]$$

i.e. can be 95% confident that the number of computers damaged lies in the range $516 < x < 942$

** Accept students' answers for confidence interval from part (a)(i) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* formulates correct confidence interval with some correct substitution.
High partial credit: (4 marks)	–	Finds one endpoint of interval only [ans. 516 <u>or</u> 942].

(iii) The assessor wishes to halve the margin of error in **part (i)** above. Assuming that the proportion of computers that are damaged remains unchanged, how many computers should he include in the random sample? **(5C)**

N = number in new sample

n = number in old sample (140)

Margin of error in new sample

$$= z\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}$$

$\hat{p}$ remains unchanged

$$\Rightarrow z\sqrt{\frac{\hat{p}(1-\hat{p})}{N}} = \frac{1}{2}\left(z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{N}} = \frac{1}{2}\left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{N}} = \frac{1}{2}\left(\frac{1}{\sqrt{140}}\right)$$

$$\Rightarrow \frac{1}{N} = \frac{1}{4}\left(\frac{1}{140}\right)$$

$$\Rightarrow N = 4(140)$$

$$= 560$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down <u>correct formula for margin of error</u>, *i.e.* $z\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}$ <u>or</u> $1{\cdot}96\sqrt{\frac{\hat{p}(1-\hat{p})}{N}}$ <u>and stops</u>.
High partial credit: (4 marks)	–	Finds $\frac{1}{\sqrt{N}} = \frac{1}{2}\left(\frac{1}{\sqrt{n}}\right)$ with correct substitution into equation, but fails to finish <u>or</u> finishes incorrectly.

Question 8 (cont'd.)

8(b) Lactate dehydrogenase (LDH) is an enzyme found in nearly all living cells. Measuring LDH levels can be helpful in monitoring the effectiveness of certain medical treatments. For a particular group of patients in a research study, the distribution of LDH levels was normal with a mean of 210 and a standard deviation of 15.

(i) Find the proportion of patients in the study with LDH levels of between 200 and 240. **(10D)**

$$z = \frac{x-\mu}{\sigma}$$

$\mu = 210$
$\sigma = 15$

$\Rightarrow z_{200} = \frac{200-210}{15} = -0{\cdot}666666... \cong -0{\cdot}67$

$\Rightarrow z_{240} = \frac{240-210}{15} = 2$

$\Rightarrow P(200 \le x \le 240) = P(-0{\cdot}67 < z < 2)$
$= P(z < 2) - P(z < -0{\cdot}67)$
$= P(z < 2) - (1 - P(z < 0{\cdot}67))$
$= 0{\cdot}9772 - (1 - 0{\cdot}7486)$ ... from z-tables
$= 0{\cdot}9772 - 0{\cdot}2514$
$= 0{\cdot}7258$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for z-value with some correct substitution.
	–	Finds correct value for either z_{200} <u>or</u> z_{240} <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Finds one z-value and related z-score, *i.e.* $P(z < 2) = 0{\cdot}9772$ <u>or</u> $P(z < 0{\cdot}67) = 0{\cdot}7486$, <u>and stops</u> <u>or</u> fails to progress.
	–	Finds correct $P(-0{\cdot}67 < z < 2)$ <u>and stops</u> <u>or</u> fails to progress (no z-scores found).
High partial credit: (8 marks)	–	Finds both z-values <u>and</u> z-scores, but fails to manipulate $P(z < -0{\cdot}67)$ correctly.
	–	Finds correct $P(z < 2) - (1 - P(z < 0{\cdot}67))$, but fails to finish <u>or</u> finishes incorrectly, *e.g.* fails to find <u>or</u> finds incorrect z-values.

(ii) Reduced levels of LDH usually indicate that the medical treatments are working. Find the lower quartile of LDH levels for patients in this research group. **(5C)**

$\Rightarrow P(z \le k) = 0{\cdot}25$
$\Rightarrow z = -0{\cdot}675$ ... from z-tables

$$z = \frac{x-\mu}{\sigma}$$

$\mu = 210$
$\sigma = 15$

$\Rightarrow -0{\cdot}675 = \frac{x-210}{15}$

$\Rightarrow x - 210 = -0{\cdot}675(15) = -10{\cdot}125$

$\Rightarrow x = 210 - 10{\cdot}125 = 199{\cdot}875$

Question 8 (cont'd.)

8(b) (ii) (cont'd.)

** Accept students' answers based on z-values of $-0{\cdot}67$ [ans. 199·95] <u>or</u> $-0{\cdot}68$ [ans. 199·8]

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* sketches graph of normal distribution with lower quartile <u>or</u> 25% indicated.
	–	Finds $z = \dfrac{x - 210}{15}$ <u>and stops</u> <u>or</u> fails to progress (no z-score found).
High partial credit: (4 marks)	–	Finds $P(z \le k) = -0{\cdot}675$ (<u>or</u> $-0{\cdot}67\,/\,-0{\cdot}68$) with correct substitution into formula for z-value, *i.e.* $-0{\cdot}675 = \dfrac{x - 210}{15}$, but fails to finish <u>or</u> finishes incorrectly.

(iii) One month later, 100 of these patients were randomly selected and underwent further testing. It was found that their LDH levels were normally distributed with a mean of 208 and the same standard deviation.
Using the sample mean, find a 95% confidence interval for the mean LDH level in this group of patients. **(5C)**

95% confidence interval for the mean LDH level in the group of patients retested (μ)

$$= \left[\bar{x} - 1{\cdot}96\frac{\sigma}{\sqrt{n}},\ \bar{x} + 1{\cdot}96\frac{\sigma}{\sqrt{n}}\right]$$

$\bar{x} = 208$
$\sigma = 15$
$n = 100$

$\Rightarrow$ 95% confidence interval

$$= \left[208 - 1{\cdot}96\left(\frac{15}{\sqrt{100}}\right),\ 208 + 1{\cdot}96\left(\frac{15}{\sqrt{100}}\right)\right]$$
$$= [\,208 - 1{\cdot}96(1{\cdot}5),\ 208 + 1{\cdot}96(1{\cdot}5)\,]$$
$$= [\,208 - 2{\cdot}94,\ 208 + 2{\cdot}94\,]$$
$$= [\,205{\cdot}06,\ 210{\cdot}94\,]$$

i.e. 95% confidence that the mean LDH level in this group of patients lies in the range $205{\cdot}06 < \mu < 210{\cdot}94$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct formula for confidence interval, *i.e.* $\bar{x} \pm z\dfrac{\sigma}{\sqrt{n}}$ <u>or</u> $\bar{x} \pm 1{\cdot}96\dfrac{\sigma}{\sqrt{n}}$, <u>and stops</u>.
	–	Some correct substitution ($\bar{x}$, σ <u>or</u> n) into 95% confidence interval (not stated) <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (4 marks)	–	Correct substitution into 95% confidence interval, but fails to finish <u>or</u> finishes incorrectly.

8(b) (cont'd.)

(iv) Test the hypothesis, at the 5% level of significance, that the mean LDH level has not changed in this period of time.
State clearly the null hypothesis and the alternative hypothesis.
Give your conclusion in the context of the question. **(5C)**

❶ $H_0 : \mu = 210$ – mean has not changed in the last month
$H_1 : \mu \neq 210$ – mean has changed in the last month

❷ 95% confidence interval for the mean LDH level in the group of patients retested (μ)
= $[\,205 \cdot 06,\ 210 \cdot 94\,]$ ... answer from part (b)(iii)

❸ <u>Conclusion</u>
as 210 is inside the interval for the mean LDH level for the population group within the research study, $205 \cdot 06 < \mu < 210 \cdot 94$, we fail to reject the null hypothesis (H_0),
i.e. conclude that the mean LDH level has not changed in that time period

** Accept students' answers for confidence interval from part (b)(iii) if not oversimplified.

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct null hypothesis <u>and/or</u> alternative hypothesis only.
High partial credit: (4 marks)	–	States both hypotheses correctly and compares population mean (μ) to the confidence interval from part (iii) but: - fails to accept <u>or</u> reject hypothesis, - fails to contextualise answer properly.

8(b) (cont'd.)

(v) Find the p-value of the test you performed in **part (iv)** above and explain what this value represents in the context of the question. **(5D)**

❶ <u>p-value</u>

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$\bar{x} = 208$
$\mu = 210$
$\sigma = 15$
$n = 100$

$$\Rightarrow z = \frac{208 - 210}{\frac{15}{\sqrt{100}}} = \frac{-2}{1 \cdot 5} = -1 \cdot 333333... \cong -1 \cdot 33$$

$$\Rightarrow P(z < -1 \cdot 33) = 1 - P(z < 1 \cdot 33) = 1 - 0 \cdot 9082 \quad \text{... from } z\text{-tables} \quad = 0 \cdot 0918$$

$$\Rightarrow p\text{-value} = 2 \times 0 \cdot 0918 = 0 \cdot 1836 > 0 \cdot 05$$

❷ <u>Explanation</u>

Any 1:

– the p-value is the probability that the test statistic or a more extreme value could occur if the null hypothesis is true //

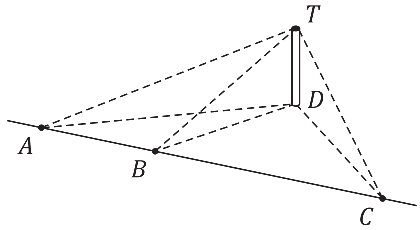
– if the mean LDH level of the population group is 210, then the probability that the mean LDH level of the sample retested would be 208 by chance is 18·36%
- it is because this has more than a 5% chance that we do not reject the null hypothesis

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct relevant formula for z-value <u>and stops</u>.
	–	Some correct substitution ($\bar{x}$, μ, σ <u>or</u> n) into formula for z-value (not stated) <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (3 marks)	–	Finds $P(z < 1 \cdot 33) = 0 \cdot 9082$, but fails to manipulate $P(z < -1 \cdot 33)$ correctly.
	–	Finds $P(z < -1 \cdot 33) = 1 - P(z < 1 \cdot 33)$, but fails to find <u>or</u> finds incorrect z-value.
High partial credit: (4 marks)	–	Finds correct p-value, but fails to contextualise answer properly.

Question 9 **(50 marks)**

Three points, A, B and C, are on a horizontal roadway such that $|AB| = 35$ m and $|BC| = 70$ m. A vertical mobile phone mast $[DT]$ has its base, D, at the same level as the roadway. The angles of elevation from A, B and C to the top of the tower, T, are such that $\tan|\angle TAD| = \frac{3}{20}$, $\tan|\angle TBD| = \frac{1}{5}$ and $\tan|\angle TCD| = \frac{3}{13}$.



9(a) (i) Let h be the height of the mobile phone mast, $|DT|$.
Express $|DA|$, $|DB|$ and $|DC|$, in terms of h. **(10D)**

Using trigonometry

❶ Consider ΔADT

$$\tan|\angle TAD| = \frac{|DT|}{|DA|} = \frac{3}{20} \text{ (given)}$$

$$|DT| = h$$

$$\Rightarrow \frac{h}{|DA|} = \frac{3}{20}$$

$$\Rightarrow |DA| = \frac{20h}{3}$$

❷ Consider ΔBDT

$$\tan|\angle TBD| = \frac{|DT|}{|DB|} = \frac{1}{5} \text{ (given)}$$

$$\Rightarrow \frac{h}{|DB|} = \frac{1}{5}$$

$$\Rightarrow |DB| = 5h$$

❸ Consider ΔCDT

$$\tan|\angle TCD| = \frac{|DT|}{|DC|} = \frac{3}{13} \text{ (given)}$$

$$\Rightarrow \frac{h}{|DC|} = \frac{3}{13}$$

$$\Rightarrow |DC| = \frac{13h}{3}$$

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* draws <u>or</u> indicates on diagram ΔADT, ΔBDT <u>or</u> ΔCDT with correct lengths of sides shown [*i.e.* opposite (h) and relevant angle].
	–	Some correct substitution into correct trig ratio (tan), *e.g.* $\tan|\angle TAD| = \frac{h}{|DA|}$, <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Finds one distance correct in terms of h.
High partial credit: (8 marks)	–	Finds two distances correct in terms of h.

Question 9 (cont'd.)

9(a) (cont'd.)

(ii) Use the cosine rule to find $\cos|\angle ABD|$ in the form $\dfrac{a - h^2}{bh}$, where $a, b \in \mathbb{N}$. **(10D)**

Using cosine rule

Consider ΔABD

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\Rightarrow |DA|^2 = |AB|^2 + |DB|^2 - 2|AB|.|DB|\cos|\angle ABD|$$
$$|DA| = \frac{20h}{3} \quad \text{... answer from part (a)(i)}$$
$$|AB| = 35 \text{ (given)}$$
$$|DB| = 5h \quad \text{... answer from part (a)(i)}$$
$$\Rightarrow \left(\frac{20h}{3}\right)^2 = (35)^2 + (5h)^2 - 2(35)(5h)\cos|\angle ABD|$$
$$\Rightarrow \frac{400h^2}{9} = 1{,}225 + 25h^2 - 350h\cos|\angle ABD|$$
$$\Rightarrow 350h\cos|\angle ABD| = 1{,}225 + 25h^2 - \frac{400h^2}{9}$$
$$= 1{,}225 - \frac{175h^2}{9}$$
$$\Rightarrow 3{,}150h\cos|\angle ABD| = 11{,}025 - 175h^2$$
$$\Rightarrow \cos|\angle ABD| = \frac{11{,}025 - 175h^2}{3{,}150h}$$
$$= \frac{63 - h^2}{18h}$$

** Accept students' answers for $|DA|$ and $|DB|$ from part (a)(i) if not oversimplified.

Scale 10D (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* draws <u>or</u> indicates on diagram ΔABD with correct lengths of sides shown [*i.e.* sides *a*, *b*, *c* and relevant angle].
	–	Some correct substitution into cosine rule <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Fully correct substitution into cosine rule <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (8 marks)	–	Fully correct substitution into cosine rule with substantive work towards isolating $\cos|\angle ABD|$, but fails to finish <u>or</u> finishes incorrectly.
	–	Isolates $\cos|\angle ABD|$ correctly, but fails to give final answer in required form.

9(a) (cont'd.)

(iii) Similarly, show that $\cos|\angle DBC| = \dfrac{1{,}575 + 2h^2}{225h}$. **(5D)**

<u>Using cosine rule</u>

Consider ΔDBC

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow |DC|^2 = |BC|^2 + |DB|^2 - 2|BC|.|DB|\cos|\angle DBC|$$

$|DC| = \dfrac{13h}{3}$ ... answer from part (a)(i)

$|BC| = 70$ (given)

$|DB| = 5h$ ... answer from part (a)(i)

$$\Rightarrow \left(\frac{13h}{3}\right)^2 = (70)^2 + (5h)^2 - 2(70)(5h)\cos|\angle DBC|$$

$$\Rightarrow \frac{169h^2}{9} = 4{,}900 + 25h^2 - 700h\cos|\angle DBC|$$

$$\Rightarrow 700h\cos|\angle DBC| = 4{,}900 + 25h^2 - \frac{169h^2}{9}$$

$$= 4{,}900 + \frac{56h^2}{9}$$

$$\Rightarrow 6{,}300h\cos|\angle DBC| = 44{,}100 + 56h^2$$

$$\Rightarrow \cos|\angle DBC| = \frac{44{,}100 + 56h^2}{6{,}300h}$$

$$= \frac{1{,}575 + 2h^2}{225h}$$

** Accept students' answers for $|DC|$ and $|DB|$ from part (a)(i) if not oversimplified.

Scale 5D (0, 2, 3, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* draws <u>or</u> indicates on diagram ΔDBC with correct lengths of sides shown [*i.e.* sides *a*, *b*, *c* and relevant angle].
	–	Some correct substitution into cosine rule <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (3 marks)	–	Fully correct substitution into cosine rule <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (4 marks)	–	Fully correct substitution into cosine rule with substantive work towards isolating $\cos|\angle DBC|$, but fails to finish <u>or</u> finishes incorrectly.
	–	Isolates $\cos|\angle DBC|$ correctly, but fails to give final answer in required form.

9(a) (cont'd.)

(iv) Show that $\cos(180° - \theta) = -\cos\theta$. **(5C)**

$$
\begin{aligned}
\cos(A - B) &= \cos A \cos B + \sin A \sin B \\
\Rightarrow \cos(180° - \theta) &= \cos 180° \cos\theta + \sin 180° \sin\theta \\
\cos 180° &= -1 \\
\sin 180° &= 0 \\
\Rightarrow \cos(180° - \theta) &= (-1)\cos\theta + (0)\sin\theta \\
&= -\cos\theta
\end{aligned}
$$

Scale 5C (0, 2, 4, 5)

Low partial credit: (2 marks)	–	Any relevant first step, *e.g.* writes down correct expansion of $\cos(A - B)$ with some correct substitution (A <u>or</u> B).
High partial credit: (4 marks)	–	Expands $\cos(180° - \theta)$ correctly and identifies $\cos 180° = -1$ and $\sin 180° = 0$, but fails to finish <u>or</u> finishes incorrectly.

(v) Hence, or otherwise, find the value of h. **(10D*)**

$$
\begin{aligned}
|\angle ABD| + |\angle DBC| &= 180° && \text{... as } ABC \text{ is a straight line} \\
\Rightarrow |\angle ABD| &= 180° - |\angle DBC| \\
\Rightarrow \cos|\angle ABD| &= \cos(180° - |\angle DBC|) \\
\cos(180° - \theta) &= -\cos\theta && \text{... from part (a)(iv)} \\
\Rightarrow \cos|\angle ABD| &= -\cos|\angle DBC| \\
\Rightarrow \frac{63 - h^2}{18h} &= -\frac{1{,}575 + 2h^2}{225h} && \text{... from parts (a)(ii) and (a)(iii)} \\
\Rightarrow 225(63 - h^2) &= -18(1{,}575 + 2h^2) \\
\Rightarrow 14{,}175 - 225h^2 &= -28{,}350 - 36h^2 \\
\Rightarrow 225h^2 - 36h^2 &= 14{,}175 + 28{,}350 \\
\Rightarrow 189h^2 &= 42{,}525 \\
\Rightarrow h^2 &= \frac{42{,}525}{189} \\
&= 225 \\
\Rightarrow h &= \sqrt{225} \\
&= 15 \text{ m}
\end{aligned}
$$

** Accept students' answers for $\cos|\angle ABD|$ from part (a)(ii) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	– –	Any relevant first step, *e.g.* writes down $|\angle ABD| + |\angle DBC| = 180°$ <u>or similar</u>. Finds $\frac{63 - h^2}{18h} = \frac{1{,}575 + 2h^2}{225h}$ (incorrect sign) <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	–	Equates $\frac{63 - h^2}{18h} = -\frac{1{,}575 + 2h^2}{225h}$ (correct sign) <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (8 marks)	–	Equates $\frac{63 - h^2}{18h} = -\frac{1{,}575 + 2h^2}{225h}$ with substantive work towards isolating h, but fails to finish <u>or</u> finishes incorrectly.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m') - apply only once in each section (a), (b), (c), *etc*. of question.

Question 9 (cont'd.)

9(b) Using your answer to **part (a)(v)**, or otherwise, find the shortest distance from the foot of the tower, D, to the roadway ABC. **(10D*)**

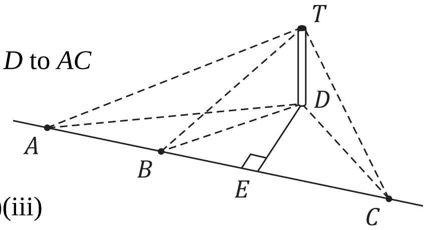
$|DB| = 5h$ ... answer from part (a)(i)

$h = 15$ ... answer from part (a)(v)

$\Rightarrow |DB| = 5(15)$

$= 75$ m

Let $|DE|$ = shortest distance ($\perp$ distance) from D to AC



$\cos|\angle DBE| = \cos|\angle DBC|$

$= \dfrac{1{,}575 + 2h^2}{225h}$

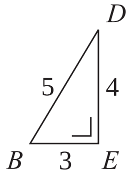
... given in part (a)(iii)

$\Rightarrow \cos|\angle DBE| = \dfrac{1{,}575 + 2(15)^2}{225(15)}$

$= \dfrac{1{,}575 + 450}{3{,}375}$

$= \dfrac{2{,}025}{3{,}375}$

$= \dfrac{3}{5}$



$\Rightarrow \sin|\angle DBE| = \dfrac{4}{5}$

$\sin|\angle DBE| = \dfrac{|DE|}{|DB|}$

$= \dfrac{|DE|}{75}$

$\Rightarrow \dfrac{|DE|}{75} = \dfrac{4}{5}$

$\Rightarrow |DE| = \dfrac{4(75)}{5}$

$= \dfrac{300}{5}$

$= 60$ m

** Accept students' answers for $|DB|$ and h from parts (a)(i) and (a)(v) if not oversimplified.

Scale 10D* (0, 4, 6, 8, 10)

Low partial credit: (4 marks)	–	Any relevant first step, *e.g.* writes down shortest distance = $\perp$ distance from D to ABC <u>or similar</u>.
	–	Finds correct value of $|DB|$ [ans. 75].
	–	Some correct substitution of h value into $\cos|\angle DBE| = \dfrac{1{,}575 + 2h^2}{225h}$ <u>and stops</u> <u>or</u> fails to progress.
Mid partial credit: (6 marks)	—	Substitutes correctly into $\cos|\angle DBE| = \dfrac{1{,}575 + 2h^2}{225h}$ <u>and</u> finds $\cos|\angle DBE| = \dfrac{3}{5}$ <u>and stops</u> <u>or</u> fails to progress.
High partial credit: (8 marks)	–	Finds $\sin|\angle DBE| = \dfrac{4}{5}$ <u>or</u> $\dfrac{|DE|}{75}$, but fails to finish <u>or</u> finishes incorrectly.

* Deduct 1 mark off correct answer only for the omission of <u>or</u> incorrect use of units ('m') - apply only once in each section (a), (b), (c), *etc.* of question.

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