L.17

NAME

SCHOOL

TEACHER

Pre-Leaving Certificate Examination, 2018

Mathematics

Paper 1

Higher Level

Time: 2 hours, 30 minutes

300 marks

School stamp

For	
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Total	

Running total

Grade

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Section A	Concepts and Skills	150 marks

Answer **all six** questions from this section.

Question 1 **(25 marks)**

(a) Solve the simultaneous equations:

$$2x + \frac{y}{2} - 2z = 8 \qquad (\times 2)$$
$$\frac{x}{2} + \frac{y}{3} + \frac{5z}{9} = 0 \qquad (\times 18)$$
$$\frac{x}{6} - \frac{y}{4} + \frac{z}{12} = \frac{7}{8}. \qquad (\times 24)$$

A: $4x + y - 4z = 16$
B: $9x + 6y + 10z = 0$
C: $4x - 6y + 2z = 21$

B + C: $9x + 6y + 10z = 0$, $4x - 6y + 2z = 21$
D: $13x + 12z = 21$

6A: $24x + 6y - 24z = 96$
C: $4x - 6y + 2z = 21$
E: $28x - 22z = 117$

11D: $143x + 132z = 231$
6E: $168x - 132z = 702$
$311x = 933$
$x = 3$

D: $13(3) + 12z = 21$
$12z = -18$
$z = -\frac{18}{12}$
$z = -\frac{3}{2}$

A: $4(3) + y - 4(-\frac{3}{2}) = 16$
$12 + y + 6 = 16$
$y = -2$

$\boxed{3, -2, -\frac{3}{2}}$

15

0, 5, 9, 12, 15

(b) If $(x + a)^2$ is a factor of $10x^3 + 21ax^2 + 20abx + 25a$, where a and b are non-zero constants, find the possible values of a and b.

$(x+a)^2 = x^2 + 2ax + a^2$

$(x^2 + 2ax + a^2)(10x + c) = 10x^3 + 21ax^2 + 20abx + 25a$

$10x^3 + cx^2 + 20ax^2 + 2acx + 10a^2x + a^2c = \text{''}$

x^3: $10 = 10$

x^2: $c + 20a = 21a$
$c = 1a$

x: $2ac + 10a^2 = 20ab$
$2a^2 + 10a^2 = 20ab$
$12a^2 = 20ab$
$\frac{12}{20}a = b$
$\frac{3}{5}a = b$

const: $a^2c = 25a$
$a^3 = 25a$
$a^2 = 5$
$\boxed{a = \pm 5}$

at $a = 5$: $b = 3$
at $a = -5$: $b = -3$

10

0, 4, 6, 8, 10

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Question 2 **(25 marks)**

(a) $z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ is a complex number, where $i^2 = -1$.

(i) Write z in polar form.

10

$r = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2} = \sqrt{1} = 1$

$\tan\theta = \frac{\sqrt{3}/2}{-1/2} = \frac{\sqrt{3}}{2} \cdot \frac{2}{-1} = -\sqrt{3}$ $\quad \cancel{A} = \pi/3$

$\theta = \frac{4\pi}{3}$

$-\frac{1}{2} - \frac{\sqrt{3}}{2} i = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$

(ii) Hence, find the four complex numbers w such that $w^4 = z$.
Give your answers in rectangular form.

$w^4 = z \qquad w = (z)^{1/4}$

$\left[\cos\left(\frac{4\pi}{3} + 2n\pi\right) + i \sin\left(\frac{4\pi}{3} + 2n\pi\right)\right]^{1/4}$

$= \cos \frac{1}{4}\left(\frac{4\pi}{3} + 2n\pi\right) + i \sin \frac{1}{4}\left(\frac{4\pi}{3} + 2n\pi\right)$

$= \cos\left(\frac{\pi}{3} + \frac{n\pi}{2}\right) + i \sin\left(\frac{\pi}{3} + \frac{n\pi}{2}\right)$

$n=0 \quad = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$n=1 \quad = \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

$n=2 \quad = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$n=3 \quad = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} i$

(b) Use De Moivre's Theorem to prove that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

$(\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)^3$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta \sin\theta\, i - 3\cos\theta \sin^2\theta - \sin^3\theta\, i$

Equate imaginary

$\sin 3\theta = 3\cos^2\theta \sin\theta - \sin^3\theta$

$= 3[1 - \sin^2\theta]\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

QED

Question 3 **(25 marks)**

(a) Solve the equation $3^{2x+2} - 28(3^x) + 3 = 0$. [Hint: Let $y = 3^x$.]

$3^{2x} . 3^2 = 9(3^x)^2$

10

0, 4, 6, 8, 10

$9y^2 - 28y + 3 = 0$

$(9y - 1)(y - 3) = 0$

$y = \frac{1}{9}$ $y = 3$

$3^x = \frac{1}{9}$

$3^x = 3^{-2}$

$x = -2$

$3^x = 3$

$x = 1$

(b) **(i)** Prove by induction that the sum of the squares of the first n natural numbers, $1^2 + 2^2 + 3^2 + \ldots + n^2$, is $\cancel{\frac{n(n+1)(n+2)}{6}}$.

$\frac{n(n+1)(2n+1)}{6}$

Show $n=1$ $\quad 1^2 = \frac{1(1+1)(2+1)}{6}$ $\quad 1 = 1$ ✓ True

assume $n=k$ $\quad 1^2 + 2^2 + 3^2 + \ldots k^2 = \frac{k(k+1)(2k+1)}{6}$

prove $n=k+1$ $\quad 1^2 + 2^2 + 3^2 + \ldots (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Proof: $\quad 1^2 + 2^2 + 3^2 + \ldots k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

10

0 4 6 8 10

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$

$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$

$= \frac{(k+1)[2k^2 + k + 6k + 6]}{6}$

$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$ True

⇒ true for $n = k+1$ ∴ True for $n = 1+1 = 2$

⇒ Since true for $n=1$, $n=2$ ∴ True for $n = 2+1 = 3$ etc

⇒ True for all $n \in N$

(ii) Hence, or otherwise, evaluate the sum of all the squares from 30 to 60, inclusive.

5

0, 2, 4, 5

$S_{60} = \frac{60(60+1)(120+1)}{6} = 73810$

$S_{29} = \frac{29(29+1)(58+1)}{6} = 8555$

$65,255$

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Question 4

(25 marks)

Dan and Kate plan to buy a house which costs €250 000. In order to get a mortgage on the property, the couple need to save a deposit of 10% of the purchase price. They open a savings account in their local Credit Union which offers an annual equivalent rate (AER) of 3·5%. $0{\cdot}035$

(a) (i) Show that the rate of interest, compounded monthly, which is equivalent to an AER of 3·5% is 0·287%, correct to three decimal places.

$(1+0{\cdot}035) = (1+i)^{12}$

$\sqrt[12]{1{\cdot}035} - 1 = i$

$0{\cdot}0028708987 19 = i$

$0{\cdot}287\% = i$

5
0, 2, 4, 5

(ii) Dan and Kate decide to put €500 in the savings account at the beginning of each month. How long will it take them to save up the deposit for the house? Give your answer in months, correct to the nearest month.

10
0, 4, 6, 8, 10.

10% of 250000 = 25,000

$500(1{\cdot}00287) + 500(1{\cdot}00287)^2 + \cdots 500(1{\cdot}00287)^n = 25{,}000$

$S_n = \dfrac{a(1-r^n)}{1-r}$

$a = 500(1{\cdot}00287)$

$r = 1{\cdot}00287$

$n = n$

$\dfrac{500(1{\cdot}00287)\left[1 - 1{\cdot}00287^n\right]}{1-1{\cdot}00287} = 25000$

$501{\cdot}435(1 - 100287^n) = -71{\cdot}75$

$1 - 1{\cdot}00287^n = -0{\cdot}1430893336$

$-1{\cdot}00287^n = -1{\cdot}1430893336$

$n \ln(1{\cdot}00287) = \ln(1{\cdot}1430893336)$

$n = 46{\cdot}66423564$

$n = 47$ months

0.00425   60 months.

(b) After saving for three years, Dan and Kate find the perfect house. They decide to borrow the remainder of the deposit at a monthly interest rate of 0·425%, fixed for the term of the loan. The loan is to be repaid in equal monthly repayments over five years and the first repayment is due one month after the loan is issued. Find the amount of each monthly repayment.

10

0, 4, 6, 8, 10

after 3 yrs (36 months) of saving.

$500(1.00287) + 500(1.00287)^2 + \cdots 500(1.00287)^{36}$.

$S_{36} = \dfrac{500(1.00287)\left[1-(1.00287)^{36}\right]}{1-1.00287}$

$= 18{,}988.51$ saved.

$25{,}000 - 18988.51 = 6011.49$ to be borrowed.

$\dfrac{x}{1.00425} + \dfrac{x}{(1.00425)^2} + \cdots \dfrac{1}{(1.00425)^{60}} = 6011.49$

S60

$\dfrac{\dfrac{x}{1.00425}\left(1-\left(\dfrac{1}{1.00425}\right)^{60}\right)}{1-\dfrac{1}{1.00425}} = 6011.49$.

$\dfrac{x}{1.00425}(53.08693925) = 6011.49$

$x(53.8693925) = \cancel{6037.038833}\ 6012.49425$

$x = \cancel{112.0680697}\ 113.2575043$

$x = \cancel{112.07}\ 113.26.$

Amortisation $A = P\dfrac{i(1+i)^t}{(1+i)^t - 1}$

$A = 6011.49\ \dfrac{(0.00425)(1.00425)^{60}}{(1.00425)^{60}-1}$

$A = 113.7198512$

$A = 113.72$.

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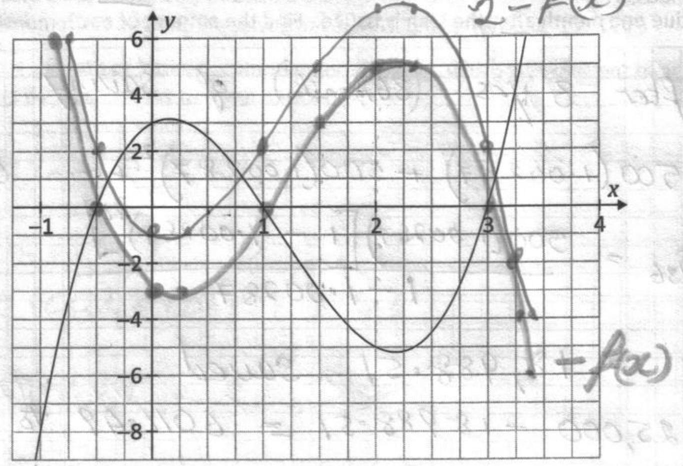
Question 5 **(25 marks)**

The diagram shows part of the graph of a cubic function $f(x)$, where $x \in \mathbb{R}$.



(a) Find the equation of $f(x)$.

10
0, 4, 6, 8, 10

$x = -\frac{1}{2}$ $\quad x = 1 \quad x = 3$

$(2x+1)(x-1)(x-3)$

$(2x^2 - 2x + x - 1)(x-3)$

$2x^3 - 6x^2 - 2x^2 + 6x + x^2 - 3x - x + 3$

$f(x) = 2x^3 - 7x^2 + 2x + 3$

Test $(0, 3)$

$2(0)^3 - 7(0)^2 + 2(0) + 3 = 3$ ✓

(b) On the diagram above, draw the graph of the function $g(x) = 2 - f(x)$, where $x \in \mathbb{R}$.

5
0, 2, 4, 5

lift up 2 → flip.

(c) Find the average value of $g(x)$ over the interval $0 \le x \le 3$, $x \in \mathbb{R}$.

10
0, 4, 6, 8, 10

$2 - (2x^3 - 7x^2 + 2x + 3)$

$g(x) = -2x^3 + 7x^2 - 2x - 1$

$\frac{1}{3-0}\int_0^3 (-2x^3 + 7x^2 - 2x - 1)\,dx = \frac{1}{3}\left[-\frac{x^4}{2} + \frac{7x^3}{3} - x^2 - x\right]_0^3$

$\frac{1}{3}\left[\left(\frac{-(3)^4}{2} + \frac{7(3)^3}{3} - (3)^2 - 3\right) - (0)\right] = \frac{1}{3}\left(\frac{21}{2}\right) = \frac{7}{2}$

Question 6 **(25 marks)**

(a) Let $f(x) = \ln\sqrt{\frac{x+1}{x-1}}$, for $x > 1$, where $x \in \mathbb{R}$.

(i) Find $f'(x)$, the derivative of $f(x)$. Give your answer in the form $\frac{a}{a - ax^2}$, where $a \in \mathbb{Z}$.

10

0, 4, 7, 10

$f(x) = \ln\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) = \frac{1}{2}\left[\ln(x+1) - \ln(x-1)\right]$

$f'(x) = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{1}{2}\left[\frac{x-1-x-1}{(x+1)(x-1)}\right]$

$= \frac{1}{2}\left[\frac{-2}{x^2-1}\right] = \frac{-2}{2x^2-2} = \frac{-1}{x^2-1} = \frac{1}{1-x^2}$

(ii) Hence, find the co-ordinates of the point at which the slope of the tangent to the curve $y = f(x)$ is parallel to the line $x + 3y - 1 = 0$. $m = -1/3$

5

0, 2, 4, 5.

$\frac{1}{1-x^2} = -1/3$

$3 = x^2 - 1$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$x = -2 \quad x = 2.$

find y. $y = \ln\sqrt{\frac{x+1}{x-1}}$

at $x = -2$ invalid since $x > 1$. ~~$y = \ln\sqrt{\frac{-2+1}{-2-1}} =$~~

at $x = 2$. $y = \ln\sqrt{\frac{2+1}{2-1}}$

$(2, \ln 3)$ or $(2, 1.098612289)$

10

0, 4, 6, 8, 10

(b) Find the co-ordinates of the point of inflection of the curve $y = \frac{xe^{x+1}}{e^{2-x}}$.

$f''(x) = 0$

$y = \frac{x e^{x+1}}{e^{2-x}} \Rightarrow x e^{2x-1}$

$dy/dx = (x)(e^{2x-1})(2) + (e^{2x-1})(1)$

$= e^{2x-1}(2x+1).$

$d^2y/dx^2 = (e^{2x-1})(2) + (2x+1)(e^{2x-1})(2)$

$= 2e^{2x-1}(1 + 2x + 1)$

$= 2e^{2x-1}(2x+2) = 0$

$2x = -2$

$x = -1$

$y = \frac{(-1)e^0}{e^3}$

$y = -\frac{1}{e^3}$

$(-1, -1/e^3)$

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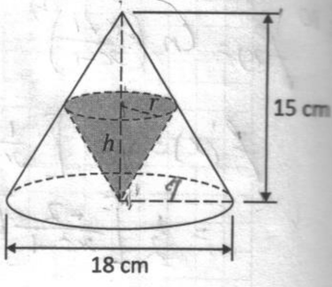
Section B Contexts and Applications 150 marks

Answer **all three** questions from this section.

Question 7 **(45 marks)**

(a) The diagram shows a right circular cone of radius 9 cm and height 15 cm. A smaller inverted cone of height h and radius r is inscribed within the larger cone.



(i) Using similar triangles, or otherwise, show that

$$h = \frac{45-5r}{3}.$$

5

0, 2, 4, 5

$\frac{9}{r} = \frac{15}{15-h}$

$h = \frac{135-15r}{9}$

$135 - 9h = 15r$

$9h = 135 - 15r$

$h = \frac{45-5r}{3}$

(ii) Express the volume of the smaller cone, in terms of π and r, in its simplest form.

5

0, 2, 4, 5

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\left(\frac{45-5r}{3}\right) = \left(\frac{45r^2 - 5r^3}{9}\right)\pi$

(iii) Find the maximum volume of the smaller cone, in terms of π.

10

0, 4, 6, 8, 10

$\frac{dV}{dr} = \frac{90r - 15r^2}{9} = 0$

$90r - 15r^2 = 0$

$15r(6-r) = 0$

$15r = 0 \quad 6 - r = 0$

$\cancel{r = 0}$ invalid $\quad r = 6\text{cm}$

~~small cone V = πr²h.~~

$V = \left(\frac{45(6)^2 - 5(6)^3}{9}\right)\pi$

$V = 60\pi \text{ cm}^3$

5

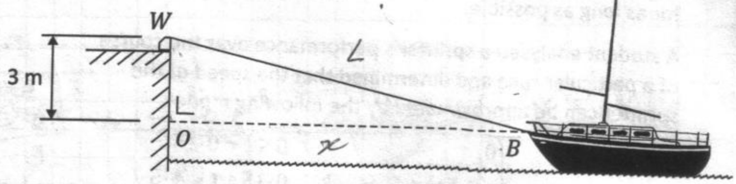
(iv) What fraction of the larger cone is unoccupied?

0, 2, 4, 5.

Vol Large Cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(9)^2(15) = 405\pi$.

$405\pi - 60\pi = 345\pi$ unoccupied $\quad \frac{345\pi}{405\pi} = \frac{23}{27}$

(b) A motorised winch is used to pull a boat into its berth position. The winch cable is attached to the bow (B) of the boat, as shown. The winch (W) is located on the quay 3 m above the bow of the boat and $|\angle WOB|$ is 90°. The winch operates at a constant speed of 0·5 m/s.



(i) Let l be the length of the winch cable, $|WB|$.
Find x, the distance of the boat from the quay wall, in terms of l.

5
0, 2, 5

$l^2 = (3)^2 + x^2$

$\sqrt{l^2 - 9} = x$

$x =$

(ii) Find the rate of change of x with respect to l.

5
0, 2, 5

$x = (l^2 - 9)^{1/2}$

$\frac{dx}{dl} = \frac{1}{2}(l^2-9)^{-1/2}(2l) = \frac{l}{\sqrt{l^2-9}}$

$\frac{dx}{dl}$

$\frac{dx}{dt}$

(iii) Hence, find the speed at which the boat is approaching the quay wall when the length of the winch cable is 13 m.

10
0, 4, 6, 8, 10

from Q $\frac{dl}{dt} = 0.5$.

$\frac{dx}{dt} = \frac{dl}{dt} \times \frac{dx}{dl}$

$= 0.5 \times \frac{l}{\sqrt{l^2-9}}$

@ $l = 13$ $= 0.5 \times \frac{13}{\sqrt{13^2-9}} = \frac{13\sqrt{10}}{80} = 0.51387 = 0.51$ m/s

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Question 8 **(55 marks)**

(a) In the 100-metre race, sprinters typically reach their top speed about halfway through the race and try to maintain that speed for as long as possible.

A student analysed a sprinter's performance over the course of a particular race and determined that the speed of the sprinter can be approximated by the following model:

$$v(t) = \begin{cases} 0, & 0 \le t < 0\cdot15 \\ -0\cdot6t^2 + 5\cdot4t - k, & 0\cdot15 \le t < 4\cdot5 \\ 11\cdot364, & t \ge 4\cdot5 \end{cases}$$

11.3535

where v is the speed in metres per second, t is the time in seconds from the starting signal and k is a constant.

(i) Find the value of k.

5

0, 2, 4, 5

$-0\cdot6t^2 + 5\cdot4t - k = \cancel{11\cdot364}\ 11\cdot3535$ @ $t = 4\cdot5$

$-0\cdot6(4\cdot5)^2 + 5\cdot4(4\cdot5) - k = 11\cdot3535$

$12\cdot15 - k = 11\cdot3535$

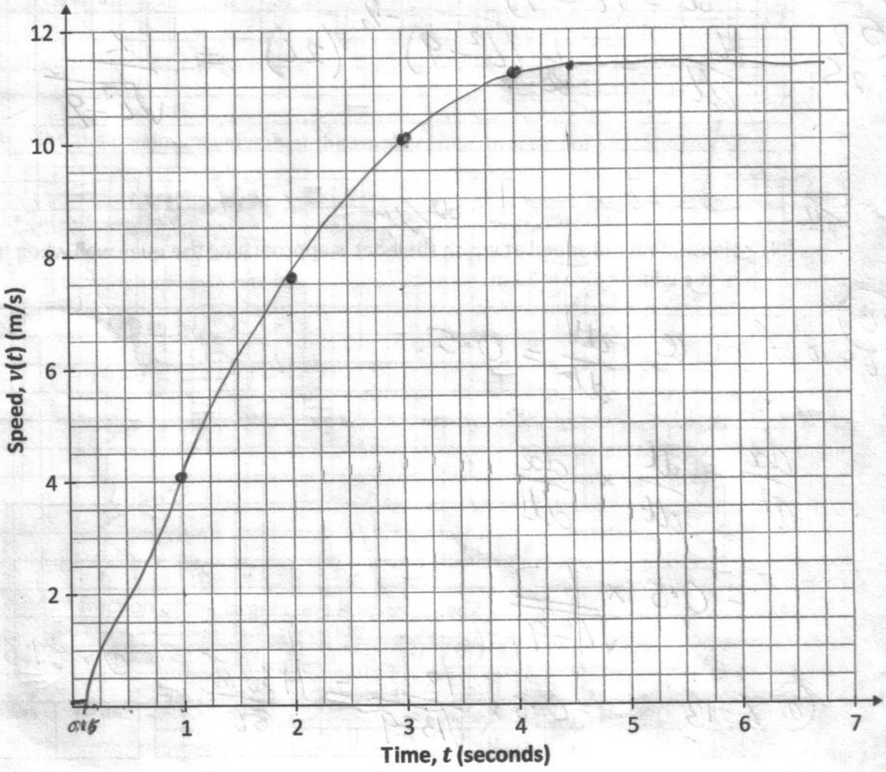
$0\cdot7965 = k$

(ii) Sketch the graph of v as a function of t for the first 7 seconds of the race.

x	fx
1	4.0035
2	7.6035
3	10.003
4	11.203
5	11.203

10

0, 4, 6, 8, 10



$0 + \int_{0.15}^{4.5}$

(iii) Find the distance travelled by the sprinter in the first 4·5 seconds of the race.

$$0 + \int_{0.15}^{4.5} (-0.6t^2 + 5.4t - 0.7965)\,dt$$

10

0, 4, 6, 8, 10

$$= \left[\frac{-0.6t^3}{3} + \frac{5.4t^2}{2} - 0.7965t\right]_{0.15}^{4.5}$$

$$\left(\frac{-0.6(4.5)^3}{3} + \frac{5.4(4.5)^2}{2} - 0.9765(4.5)\right) - \left(\frac{-0.6(0.15)^3}{3} + \frac{5.4(0.15)^2}{2} - 0.7965(0.15)\right)$$

$$32.05575 - (-0.0864)$$

$$32.14215$$

(iv) Hence, find the sprinter's finishing time for the race. Give your answer correct to three decimal places.

5

0, 2, 4, 5

$0 < t < 0.15 \Rightarrow 0\text{ m}$

$0.15 < t < 4.5 \Rightarrow 32.14215\text{ m}.$

Race $= 100\text{ m}$ $\Rightarrow 67.85785$ m to go at speed 11.364

$$\frac{67.85785}{11.364} = 5.971299718 \text{ sec}$$

P ST

$\Rightarrow$ Total time $= 4.5 + 5.97$

$= 10.47$ sec

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(b) A model for an Olympic-standard 100 m sprinter was developed by mathematicians. The velocity of the sprinter may be calculated using the function:

$$w(t) = 11 \cdot 7(1 - e^{-0 \cdot 8t}) + 0 \cdot 03(1 - e^{0 \cdot 3t})$$

where t is the time in seconds from the starting signal.

10
0, 4, 6, 8, 10

(i) Find the maximum speed of the sprinter, correct to two decimal places.

$11 \cdot 7 - 11 \cdot 7e^{-0 \cdot 8t} + 0 \cdot 03 - 0 \cdot 03e^{0 \cdot 3t}$

$\frac{dw}{dt}$ $11 \cdot 73 - 11 \cdot 7e^{-0 \cdot 8t} - 0 \cdot 03e^{0 \cdot 3t}$

$\frac{dw}{dt} = -11 \cdot 7e^{-0 \cdot 8t}(-0 \cdot 8) - 0 \cdot 03e^{0 \cdot 3t}(0 \cdot 3)$

$9 \cdot 384e^{-0 \cdot 8t} - 0 \cdot 009e^{0 \cdot 3t} = 0$

$\frac{9 \cdot 384}{e^{0 \cdot 8t}} = 0 \cdot 009e^{0 \cdot 3t}$

$9 \cdot 384 = 0 \cdot 009e^{1 \cdot 1t}$

$1042 \cdot \dot{6} = e^{1 \cdot 1t}$

$\ln 1042 \cdot \dot{6} = 1 \cdot 1t(\ln e)$

$6 \cdot 31776 = t$

$\Rightarrow w = 11 \cdot 45568...$

$w = 11 \cdot 46.$

10
0, 4, 7, 10

(ii) Find an expression for the distance travelled by the sprinter after time t.

$dis = \int w(t)\,dt$

$\int (11 \cdot 73 - 11 \cdot 7e^{-0 \cdot 8t} - 0 \cdot 03e^{0 \cdot 3t})\,dt$

$= 11 \cdot 73t - \frac{11 \cdot 7e^{-0 \cdot 8t}}{-0 \cdot 8} - \frac{0 \cdot 03e^{0 \cdot 3t}}{0 \cdot 3} + C$

$11 \cdot 73t + 14 \cdot 625e^{-0 \cdot 8t} - 0 \cdot 1e^{0 \cdot 3t} + C$

at $t=0$ $S=0$

$0 + 14 \cdot 625 - 0 \cdot 1 + C = 0$

$C = -14 \cdot 525.$

$\Rightarrow dis(S) = 11 \cdot 73t + 14 \cdot 625e^{-0 \cdot 8t} - 0 \cdot 1e^{0 \cdot 3t} - 14 \cdot 525.$

5
0, 2, 3, 4, 5

(iii) Hence, show that the sprinter completes the race in less than 10 seconds.

dis after 10 sec.

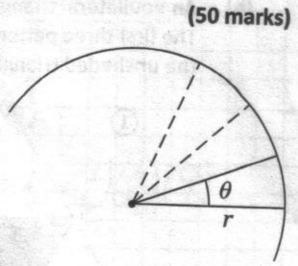
$= 11 \cdot 73(10) + 14 \cdot 625e^{-0 \cdot 8(10)} - 0 \cdot 1e^{0 \cdot 3(10)} - 14 \cdot 525$

$= 100 \cdot 7713524$ m

$\Rightarrow$ completed 100m in less than 10s

Question 9 **(50 marks)**

(a) A circular disc is divided into 12 unequal sectors whose areas are in arithmetic sequence. The area of the largest sector is twice that of the smallest sector. The radius of the disc is r and the acute angle in the smallest sector is θ, in degrees, as shown. The increase in angle in subsequent sectors is λ.



(i) Find the areas of the smallest and the largest sectors, in terms of r and θ.

5

0, 2, 5

Smallest sector:	Largest sector:
$A = \frac{\theta}{360}(\pi r^2) = \frac{\theta \pi r^2}{360}$	$A = \frac{2\theta}{360}(\pi r^2) = \frac{\theta \pi r^2}{180}$

(ii) Find an expression for the acute angle of the nth sector in the arithmetic sequence and hence, write down the size of the angle in the largest sector in terms of θ and λ.

5

0, 2, 4, 5

$\theta + (\theta + \lambda) + (\theta + 2\lambda)$

$T_n = a + (n-1)d \quad a = \theta \quad d = \lambda$

$T_n = \theta + (n-1)\lambda$

$T_{12} = \cancel{\theta + 11\lambda} \ \theta + 11\lambda$

(iii) Find an equation for the sum of the acute angles in all of the sectors, in terms of θ and λ.

5

0, 2, 4, 5

$S_n = \frac{n}{2}\{2a + (n-1)d\}$

$S_{12} = 6(2\theta + 11\lambda) = 12\theta + 66\lambda$

(iv) Use your answers to **parts (ii)** and **(iii)** above to find, in degrees, the value of θ.

5

0, 2, 4, 5.

$T_{12} = \theta + 11\lambda = 2\theta$

$11\lambda = \theta$

$\lambda = \frac{\theta}{11}$

$S_{12} = 12\theta + 66\lambda = 12\theta + 66\left(\frac{\theta}{11}\right) = 360^\circ$

$18\theta = 360^\circ$

$\theta = 20^\circ$

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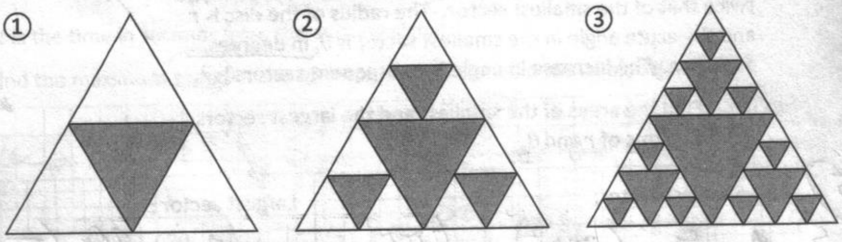
(b) An equilateral triangle can be subdivided into four smaller equilateral triangles of equal area. *The first three patterns in a sequence of patterns are shown below. In each successive pattern,* the unshaded triangle is subdivided into smaller equal triangles.



(i) Complete the table below to show the number of shaded and unshaded equilateral triangles in each pattern.

5

0, 2, 4, 5.

Pattern	1	2	3	4	5
Number of shaded triangles	1	4	13	40	121
Number of unshaded triangles	3	9	27	81	243

(ii) Write an expression in *n* for the number of unshaded triangles in the *n*th pattern in the sequence.

5

0, 2, 4, 5

$T_n = ar^{n-1}$

$T_n = 3^1(3)^{n-1} = 3^n.$

$T_n = 3^n$

(iii) Find an expression, in *n*, for the number of shaded triangles in the *n*th pattern in the sequence.

10

0, 4, 6, 8, 10

1, 4, 13, 40, 121

3, 9, 27, 81

6, 18

$T_1 = 3^0 = 1$

$T_2 = 1 + 3 = T_1 + 3^1$

$T_3 = T_2 + 3^2$

$T_4 = T_3 + 3^3$

$T_n = T_{n-1} + 3^{n-1}$

$\boxed{T_n = \dfrac{3^n - 1}{2}}$

(iv) Find the fraction of the overall area that is shaded in the 5th pattern.

5

0, 2, 4, 5

Shaded Area in $1^{st}\ \Delta = \frac{1}{4}$

" " " $2^{nd}\ \Delta = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right)(3)$

" " " $3^{rd}\ \Delta = \frac{1}{4} + \frac{1}{4}\left(\frac{1}{4}\right)3 + \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)(9)$

Pattern $\quad \frac{1}{4}, \quad \frac{1}{4} + \frac{3}{16}, \quad \frac{1}{4} + \frac{3}{16} + \frac{9}{64}, \ ---$

4th $\quad \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256}$

5th $\quad \frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \frac{27}{256} + \frac{81}{1024} = \frac{781}{1024}$.

(v) In which pattern will the shaded area be greater than 95% of the overall area?

5

0, 2, 4, 5.

$S_n = \frac{a(1-r^n)}{1-r} \qquad a = \frac{1}{4} \quad r = \frac{3}{4}$.

$S_n = \frac{\cancel{\frac{1}{4}}\left(1-\frac{3}{4}^n\right)}{\cancel{1-\frac{3}{4}}} = 0 \cdot 95$

$1 - \frac{3}{4}^n = 0 \cdot 95$

$0 \cdot 05 = \frac{3}{4}^n$

$\ln(0 \cdot 05) = n \ln\left(\frac{3}{4}\right)$

$\frac{\ln(0 \cdot 05)}{\ln\left(\frac{3}{4}\right)} = n$

$10 \cdot 4133 = n$

$\Rightarrow$ Pattern 11.

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