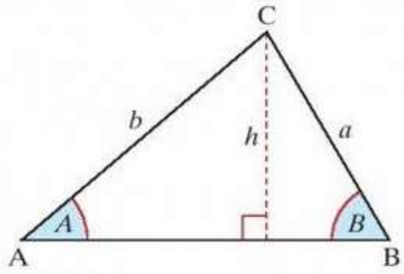
1. The Sine Rule: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Proof

Construct a perpendicular h from C to [AB].



(i) $\dfrac{h}{b} = \sin A \quad \Rightarrow \quad h = b \sin A$

(ii) $\dfrac{h}{a} = \sin B \quad \Rightarrow \quad h = a \sin B$

$\Rightarrow \quad a \sin B = b \sin A$

Divide both sides by $\sin A \sin B$

$\Rightarrow \quad \dfrac{a \sin B}{\sin A \sin B} = \dfrac{b \sin A}{\sin A \sin B}$

$\Rightarrow \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Similarly it may be shown that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Sine Rule $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

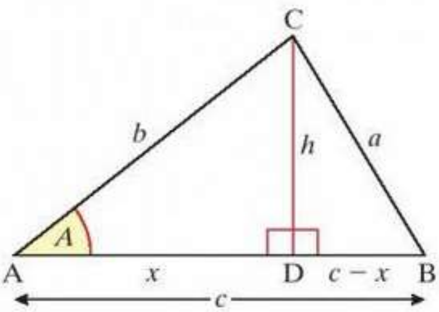
Note: If we need to find an unknown angle using the sine rule, then $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ is a more suitable form of the rule.

Using the sine rule To use the sine rule to solve a triangle, we need to know one side and the angle opposite this side as well as one other angle or side.

The Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$

***Proof**

In the △ABC, CD is perpendicular to AB.



Let $|CD| = h$ and $|AD| = x$

Thus $|DB| = c - x$

We now apply the Theorem of Pythagoras to the triangles ACD and CDB.

In △ACD: $h^2 + x^2 = b^2 \quad \Rightarrow \quad h^2 = b^2 - x^2$

In △BCD: $h^2 + (c - x)^2 = a^2 \quad \Rightarrow \quad h^2 = a^2 - (c - x)^2$

Equating the two values for h^2, we get

$a^2 - (c - x)^2 = b^2 - x^2$

$\Rightarrow \quad a^2 - (c^2 - 2cx + x^2) = b^2 - x^2$

$\Rightarrow \quad a^2 - c^2 + 2cx - x^2 = b^2 - x^2$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2cx \qquad$ But $\dfrac{x}{b} = \cos A$

$\Rightarrow \quad a^2 = b^2 + c^2 - 2c\,(b \cos A) \qquad \Rightarrow \quad x = b \cos A$

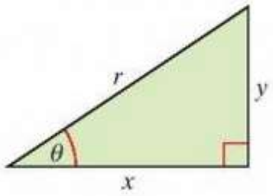
$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc \cos A$

Similarly is may be proved that

$b^2 = c^2 + a^2 - 2ca \cos B$ and $c^2 = a^2 + b^2 - 2ab \cos C$

In the given triangle

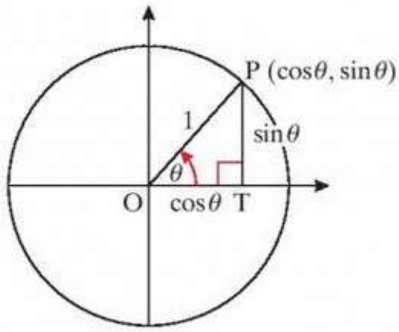
$$\frac{\sin\theta}{\cos\theta} = \frac{y}{r} \div \frac{x}{r} = \frac{y}{r} \times \frac{r}{x} = \frac{y}{x} = \tan\theta$$



$$\Rightarrow \quad \boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

This relationship between trigonometric ratios is called an **identity** because it is true for **all values** of θ.

We have already established that any point on the **unit circle** is defined by the coordinates (**cos** $\boldsymbol{\theta}$, **sin** $\boldsymbol{\theta}$).



In the given diagram $|OP| = 1$

$\Rightarrow \quad |OP|^2 = 1$

$\Rightarrow \quad \sqrt{(\cos\theta - 0)^2 + (\sin\theta - 0)^2} = 1$

$\Rightarrow \quad \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\Rightarrow \quad \boldsymbol{\cos^2\theta + \sin^2\theta = 1}$ … (squaring both sides) *

If each term in the equation $\sin^2\theta + \cos^2\theta = 1$ is divided by $\cos^2\theta$, we get,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

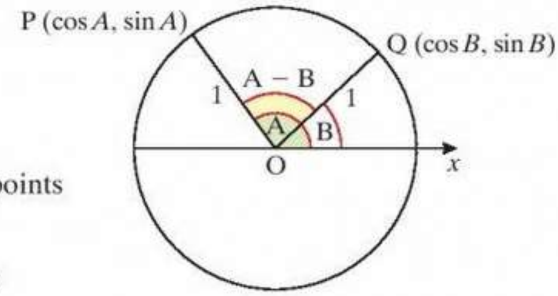
$\Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \quad \boldsymbol{1 + \tan^2\theta = \sec^2\theta}$

1. $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ **2.** $\sec\theta = \frac{1}{\cos\theta}$ **3.** $\tan\theta = \frac{\sin\theta}{\cos\theta}$

4. $\cot\theta = \frac{\cos\theta}{\sin\theta}$ **5.** $\sin^2\theta + \cos^2\theta = 1$ **6.** $1 + \tan^2\theta = \sec^2\theta$

Formula for cos(A − B)*



Let the radii [OP] and [OQ] make angles A and B with the positive x-axis.

$|\angle POQ| = A - B$

We now find $|PQ|$ using two different methods:

1. the standard formula for the distance between two points
2. the cosine rule.

1. Using the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have:

$$|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$

But $\sin^2 A + \cos^2 A = 1$ and $\sin^2 B + \cos^2 B = 1$.

$$\Rightarrow \quad |PQ|^2 = 1 - 2\cos A\cos B + 1 - 2\sin A \sin B$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B) \ldots ①$$

2. Now using the cosine rule to find $|PQ|$, we have:

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP|\,|OQ| \cos(A - B)$$
$$= 1 + 1 - 2(1)(1)\cos(A - B) \ldots \quad |OP| = |OQ| = 1 = \text{radius}$$
$$= 2 - 2\cos(A - B) \ldots ②$$

Equating the two values for $|PQ|^2$, we have

$$2 - 2(\cos A \cos B + \sin A \sin B) = 2 - 2\cos(A - B)$$
$$\Rightarrow \quad -(\cos A \cos B + \sin A \sin B) = -\cos(A - B)$$

*$\Rightarrow$ **$\cos(A - B) = \cos A \cos B + \sin A \sin B$ ...... (i)**

Having established the formula for $\cos(A - B)$, we can derive the other formulae using the important identities shown on the right.

$\cos(-B) = \cos B$
$\sin(-B) = -\sin B$
$\sin(90° - A) = \cos A$
$\cos(90° - A) = \sin A$

Formula for cos(A + B)*

To derive the formula for $\cos(A + B)$, we replace B with $(-B)$ in formula **(i)** on the previous page:

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\Rightarrow \quad \cos[A - (-B)] = \cos A \cos(-B) + \sin A \sin(-B)$$
$$= \cos A \cos B + \sin A(-\sin B)$$

*$\Rightarrow$ **$\cos(A + B) = \cos A \cos B - \sin A \sin B$ ...... (ii)**

Formulae for sin(A + B) and sin(A − B)

$\cos(A - B) = \cos A \cos B + \sin A \sin B$ ... from **(i)**

To derive the formula for $\sin(A + B)$, we replace A with $(90° - A)$.

$$\Rightarrow \quad \cos(A - B) = \cos[(90° - A) - B] = \cos(90° - A)\cos B + \sin(90° - A)\sin B$$
$$= \sin A \cos B + \cos A \sin B$$
$$\Rightarrow \quad \cos[90° - (A + B)] = \sin A \cos B + \cos A \sin B$$

*$\Rightarrow$ **$\sin(A + B) = \sin A \cos B + \cos A \sin B$ ...... (iii)**

Substituting $(-B)$ for B in formula **(iii)**, we get

$$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$\sin(-B) = -\sin B$

*$\Rightarrow$ **$\sin(A - B) = \sin A \cos B - \cos A \sin B$ ...... (iv)**

Formulae for tan(A + B) and tan(A − B)

Expressions for $\tan(A+B)$ and $\tan(A-B)$ can be derived from formulae **(i)** to **(iv)** established above:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

We now divide each term in the numerator and denominator by $\cos A \cos B$.

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}}$$

$$*\Rightarrow \quad \mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}} \text{ ...... } \mathbf{(v)}$$

Substituting $(-B)$ for B in formula **(v)** we get:

$$\mathbf{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}} \text{ ...... } \mathbf{(vi)}$$

$\tan(-B) = -\tan B$

Compound angle formulae

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$; $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Replacing B with A we get:

$$\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$$

$$\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}$$

Using the identity $\sin^2 A + \cos^2 A = 1$, we get two further identities for $\cos 2A$:

$$\cos 2A = \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A$$

or

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = \cos^2 A - 1 + \cos^2 A = 2\cos^2 A - 1$$

Double angle formulae

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\sin 2A = 2 \sin A \cos A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

By rearranging the formulae for $\cos 2A$, we can express $\cos^2 A$ and $\sin^2 A$ in terms of $\cos 2A$ as follows:

$$\cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = 1 + \cos 2A \Rightarrow \cos^2 A = \tfrac{1}{2}(1 + \cos 2A)$$

or

$$\cos 2A = 1 - 2\sin^2 A \Rightarrow 2\sin^2 A = 1 - \cos 2A \Rightarrow \sin^2 A = \tfrac{1}{2}(1 - \cos 2A)$$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$ $\qquad$ $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

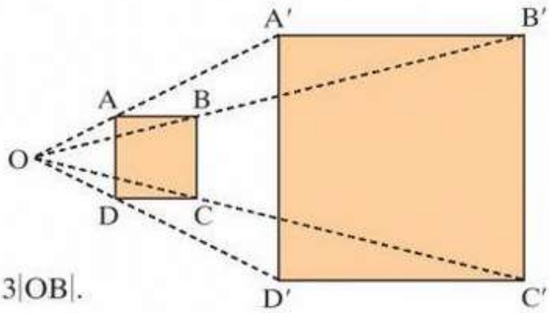
Drawing enlargements

To construct the image of a given figure under an enlargement, we need

(i) the centre of enlargement

(ii) the scale factor of the enlargement.

The diagram on the right shows a square ABCD and a centre of enlargement O.



We will now enlarge ABCD with O as centre of enlargement and scale factor 3.

To find the image of A, we join O to A and continue to A′ so that $|OA'| = 3|OA|$.

Similarly, join O to B and continue to B′ so that $|OB'| = 3|OB|$.

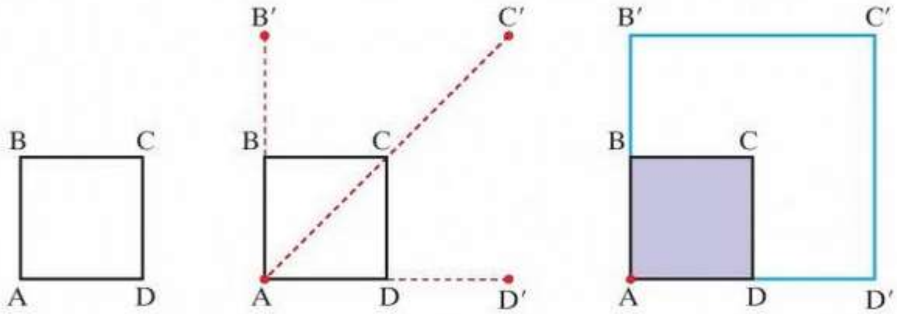
Repeat the process for the points C and D.

The square A′B′C′D′ is the image of the square ABCD.

Since the scale factor is 3, $|A'B'| = 3|AB|$ and $|A'D'| = 3|AD|$.

When a vertex is the centre of enlargement

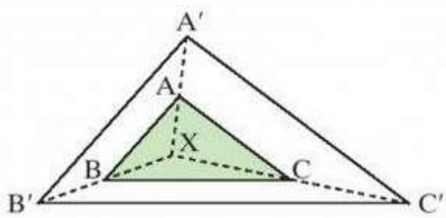
The diagrams below show how to enlarge the shape ABCD by a scale factor of 2 using A as the centre of enlargement.



Notice that the centre of enlargement, A, does not move.

In the final figure, $|AB'| = 2|AB|$, $|AD'| = 2|AD|$ and $|AC'| = 2|AC|$.

The diagram on the right shows an enlargement where the centre of enlargement, X, is inside the figure.



In this enlargement, the scale factor is 2.

Draw the line [XA] and extend it so that $|XA'| = 2|XA|$.

Extend [XB] so that $|XB'| = 2|XB|$.

Repeat for [XC].

Each side of the enlarged triangle A′B′C′ is twice the length of the corresponding side in ABC.

For any enlargement, the scale factor is found by dividing the length of the image side by the length of the corresponding object side.

The scale factor is
$$\frac{\text{length of image side}}{\text{length of corresponding object side}}$$

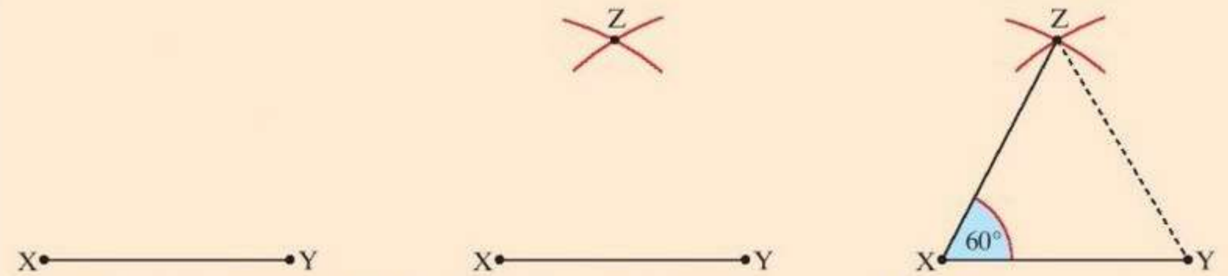
CONSTRUCTIONS:

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l, passing through a given point not on l.
4. Line perpendicular to a given line l, passing through a given point on l.
5. Line parallel to given line, through given point.
6. Division of a segment into 2, 3 equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight-edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of $60°$, without using a protractor or set square.
19. Tangent to a given circle at a given point on it.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.

1. Constructing an angle of 60°

Each angle in an equilateral triangle is 60°. We will now use this information to draw an angle of 60°.

In an equilateral triangle, all the sides are equal in length.



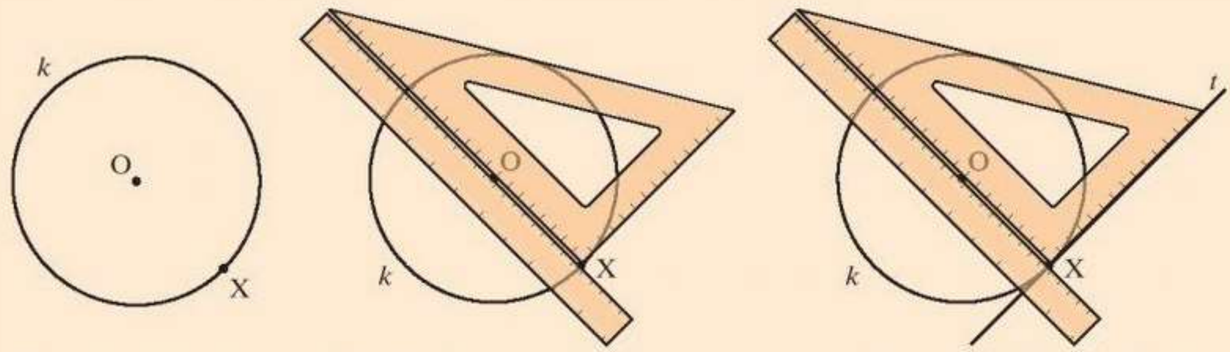
Draw a line segment [XY].

Set the compass to a radius of |XY|.
With X as centre and radius |XY| draw an arc.
Repeat at Y.
The arcs meet at Z.

Join XZ.
|∠ZXY| = 60°.

The triangle XYZ is equilateral.

2. How to construct a tangent to a circle at a given point on it



Given a circle k and a point X on it. O is the centre of the circle.

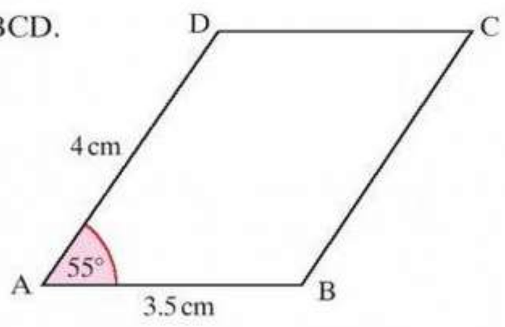
Join X to O, the centre of the circle. Place a ruler along OX and slide set square along the ruler until it reaches X.

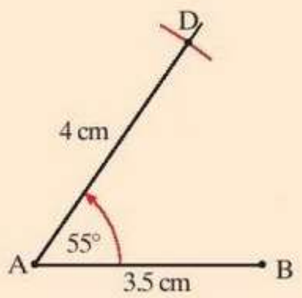
Draw a line *t* through X perpendicular to OX.
t is a tangent to the circle k.

3. How to construct a parallelogram, given the lengths of the sides and the measures of the angles

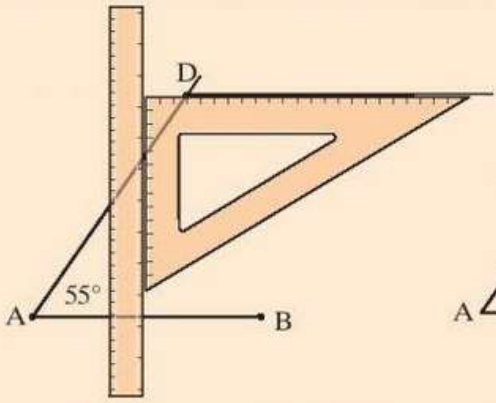
The instructions below show how to construct a parallelogram ABCD where |AB| = 3.5 cm, |AD| = 4 cm and |∠DAB| = 55°.

We first draw a rough sketch of ABCD.

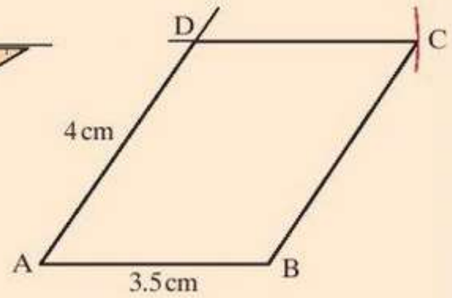




Draw a horizontal line [AB] = 3.5 cm. Use a protractor to measure an angle of 55° at A. Draw a line through A and measure |AD| = 4 cm.

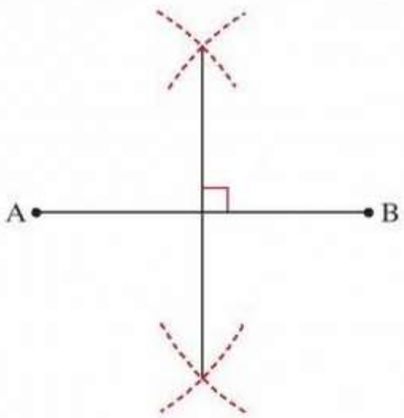


Place set square along the line AB. Use a ruler to slide the set square up to the point D. Draw a line through D parallel to AB.

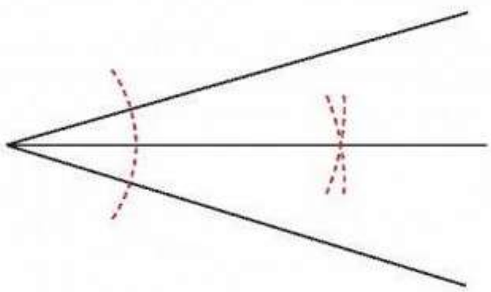


Use a compass with a radius of 3.5 cm (the same as |AB|) to draw an arc on the line. |DC| = 3.5 cm. Join BC. ABCD is the required parallelogram.

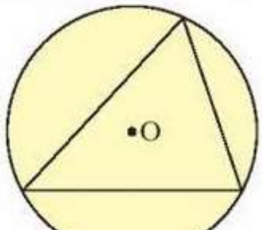
The perpendicular bisector of a line segment



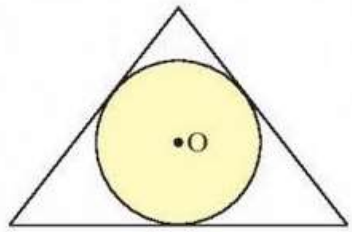
The bisector of an angle



Circles and triangles

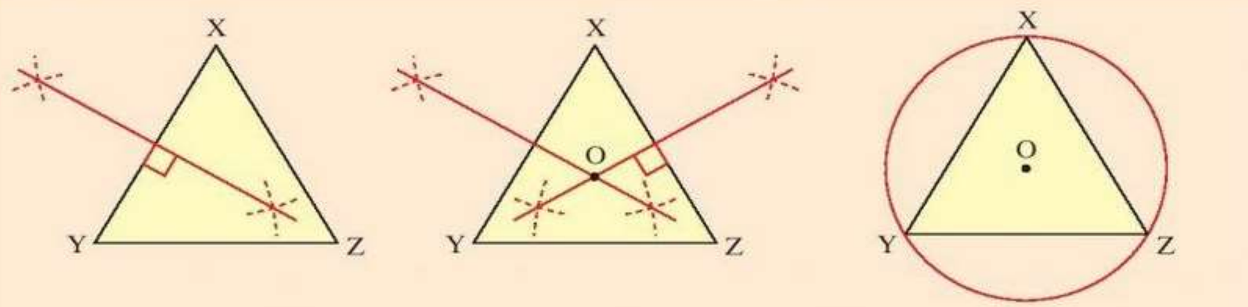


The **circumcircle** of a triangle is the circle which passes through the three vertices, as shown. The centre, O, of this circle is called the **circumcentre** of the triangle.



A circle inscribed in a triangle such that all three sides touch the circle is called the **incircle** of the triangle. The centre of the incircle is called the **incentre** of the triangle. In the figure above, O is the incentre.

4. How to construct the circumcircle of a given triangle



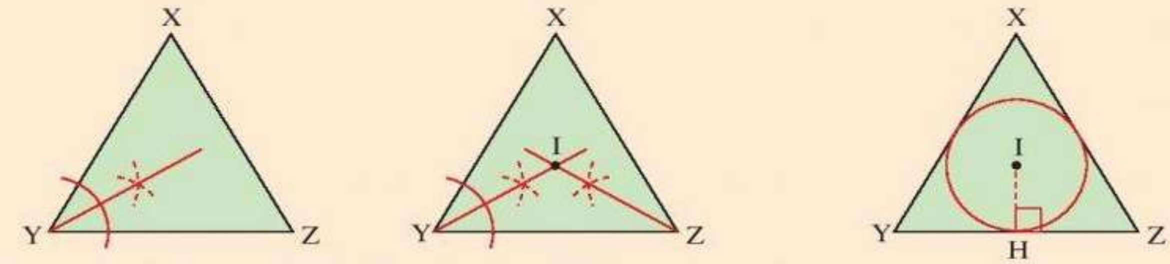
Construct the perpendicular bisector of [XY].

Construct the perpendicular bisector of [XZ]. The two bisectors meet at the point O, as shown.
O is the circumcentre.

With O as centre and |OX| as radius, draw a circle through X, Y and Z. This is the circumcircle of the triangle.

5. How to construct the incircle of a given triangle

The construction of the incircle of a triangle involves constructing the bisector of an angle which is given on the previous page.



Construct the bisector of ∠XYZ, as shown.

Now construct the bisector of ∠XZY. The two bisectors meet at the point I.
I is the incentre.

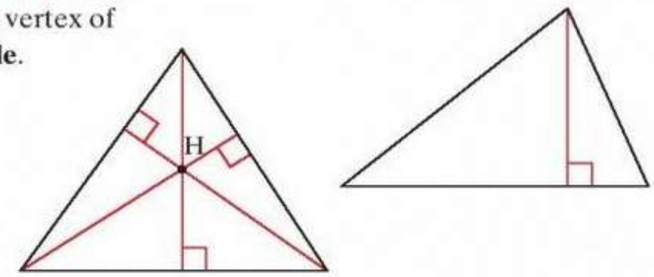
Use a set square to draw a perpendicular from I to the line YZ. The perpendicular meets YZ at H. With |IH| as radius, draw a circle to touch the three sides.
This is the incircle of the triangle XYZ.

7. How to construct the orthocentre of a triangle

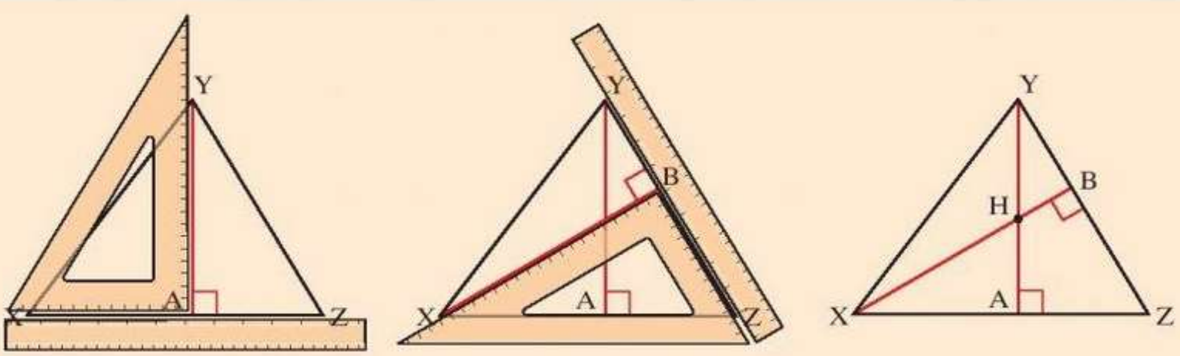
The perpendicular line segment drawn from the vertex of a triangle to the opposite side is called an **altitude**.

The point of intersection of the three altitudes of a triangle is called the **orthocentre**.

The point H is called the orthocentre.



The diagrams below illustrate the steps involved in the construction of the orthocentre of a triangle.



Place the ruler along XZ and use the set square to draw the line YA perpendicular to XZ.

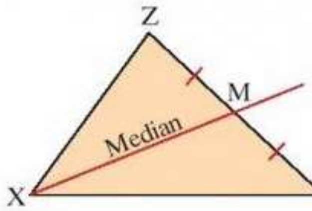
Use the ruler and set square to draw the line XB perpendicular to YZ.

The lines YA and XB meet at the point H. H is the orthocentre of the triangle.

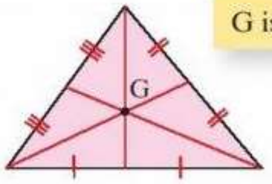
6. How to construct the centroid of a triangle

The line segment joining the vertex of a triangle to the midpoint of the opposite side is called a **median**.

In the triangle below, [XM] is a median.



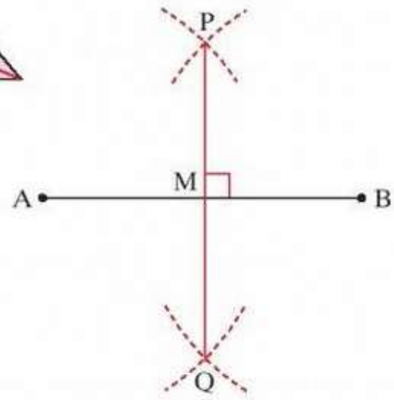
The point of intersection of the three medians of a triangle is called the **centroid** of the triangle.



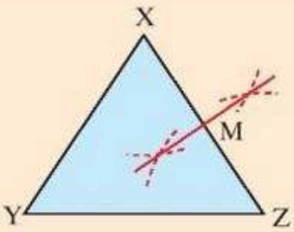
G is the centroid.

To find the midpoint of any line segment, we construct the perpendicular bisector of that line segment.

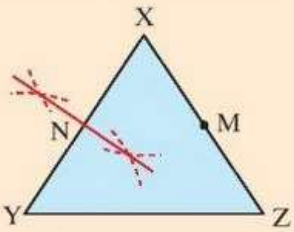
The diagram on the right should help you recall the steps involved in this construction.



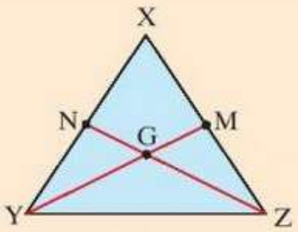
The three diagrams below illustrate the steps to be followed in the construction of the centroid of a triangle.



Construct the perpendicular bisector of [XZ], as shown. M is the midpoint of [XZ].

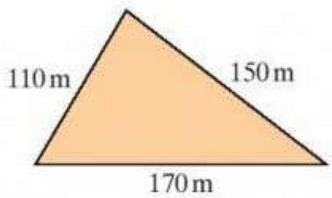


Now construct the perpendicular bisector of [XY]. N is the midpoint of [XY].

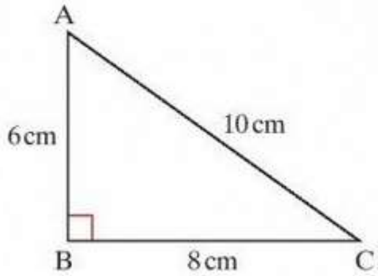


Join YM and ZN. They meet at the point G. G is the centroid of the triangle.

12. A campsite is in the shape of a triangle with busy roads running along all three sides of the site.
The sides of the site are 110 m, 150 m and 170 m in length.
 (i) Using 20 m = 1 cm, draw a scaled diagram of this site.
 (ii) Show on the diagram the best position to pitch a tent so that it is as far away as possible from all three roads. Show your construction lines.

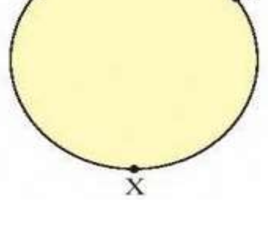


13. ABC is a right-angled triangle, as shown.
 (i) Write down the radius of the circumcircle without drawing and measuring it.
 (ii) Explain why the point B is the orthocentre of △ABC.



18. Construct an equilateral triangle of side 5 cm.
 (i) Now construct the circumcentre, O, of the triangle.
 (ii) Explain why O is also the incentre of the triangle.

17. In Q16 we used two chords to find the centre of a circle.
The given diagram shows a circle and two points on the circle.
Describe another way of finding the centre of this circle using the points X and Y.



An **axiom** is a statement accepted without proof. The angles in a straight line add to 180° is an example of an axiom.

A **theorem** is a statement that can be shown to be true through the use of axioms and logical argument.

A **corollary** is a statement attached to a theorem which has been proven and follows obviously from it.

Proof by Contradiction:

In a proof by contradiction, we show that if we claim some statement to be true, then a logical contradiction occurs which proves that the original statement must be false.

Proof:

In mathematics, we use deductive reasoning to prove that a statement is *always* true.

Converse: For statement : 'If A is true then B is true', the converse of the statement would be the statement 'If B is true then A'.

True Converse:' *In an isosceles triangle the angles opposite the equal sides are equal'* → *Converse: 'If two angles of a triangle is equal then the sides opposite the angles are equal – triangle is isosceles' ** TRUE ***

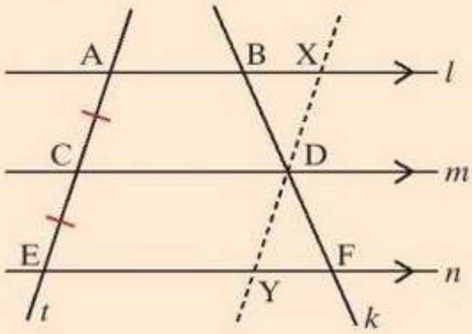
False converse: *'If two triangles are congruest then their Areas are equal'*→ *Converse: 'If two triangles have equal Area then the are congruent ** FALSE ***

Is Equivalent to: If A implies B and B implies A then A and B are said to be equivalent

If and only if : A is true if and only if B is true means A is true if B is true and B is true only if A is true

Implies: A implies B means A is true only if B is true

***Theorem 11** If three parallel lines make segments of equal length on a transversal, then they will also make segments of equal length on any other transversal.



Given: Three parallel lines l, m and n intersecting the transversal t at the points A, C and E such that $|AC| = |CE|$.
Another transversal k intersects the lines at B, D and F.

To prove: $|BD| = |DF|$.

Construction: Through D draw a line parallel to t intersecting l at X and n at Y.

Proof: ACDX and CEYD are parallelograms.

$\Rightarrow$ $|AC| = |XD|$ and $|CE| = |DY|$ ...opposite sides

But $|AC| = |CE|$.

$\Rightarrow$ $|XD| = |DY|$

In the triangles BDX and YDF,

$|XD| = |DY|$

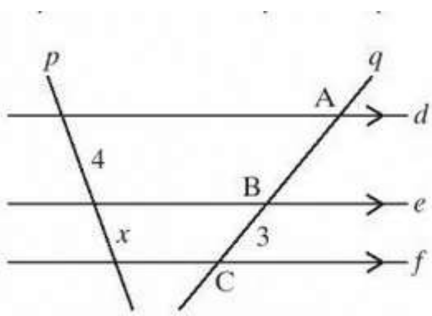
$|\angle BDX| = |\angle YDF|$ ...vertically opposite

$|\angle DBX| = |\angle DFY|$ ...alternate angles

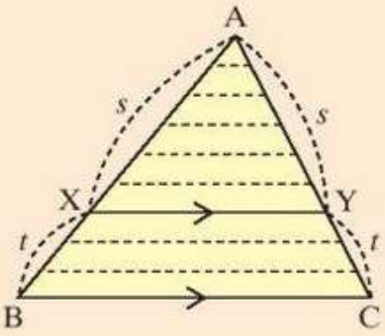
$\Rightarrow$ the triangles BDX and YDF are congruent

$\Rightarrow$ $|BD| = |DF|$ ...corresponding sides

8. In the given figure, d, e and f are parallel lines.
p and q are two transversals.
The transversal p is divided in the ratio $4:x$.
Find, in terms of x, the length of the line segment [AB].



***Theorem 12** Let ABC be a triangle. If a line XY is parallel to BC and cuts [AB] in the ratio $s:t$, then it cuts [AC] also in the same ratio.



Given: The triangle ABC with XY parallel to BC.

To Prove: $\frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

Construction: Divide [AX] into s equal parts and [XB] into t equal parts.
Draw a line parallel to BC through each point of the division.

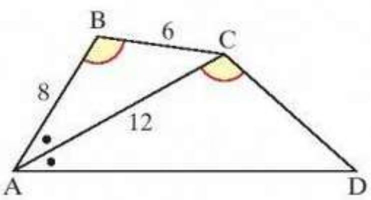
Proof: The parallel lines make intercepts of equal length along the line [AC].

$\therefore$ [AY] is divided into s equal intercepts and [YC] is divided into t equal intercepts.

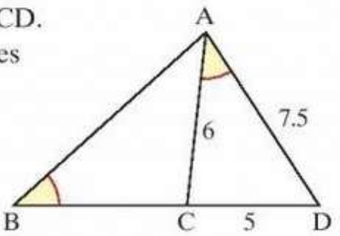
$\therefore \quad \frac{|AY|}{|YC|} = \frac{s}{t}$

But $\frac{|AX|}{|XB|} = \frac{s}{t} \quad \Rightarrow \quad \frac{|AX|}{|XB|} = \frac{|AY|}{|YC|}$

13. In the given figure, the diagonal [AC] bisects the angle BAD.
$|\angle ABC| = |\angle ACD|$.
Find (i) $|CD|$
(ii) $|AD|$.

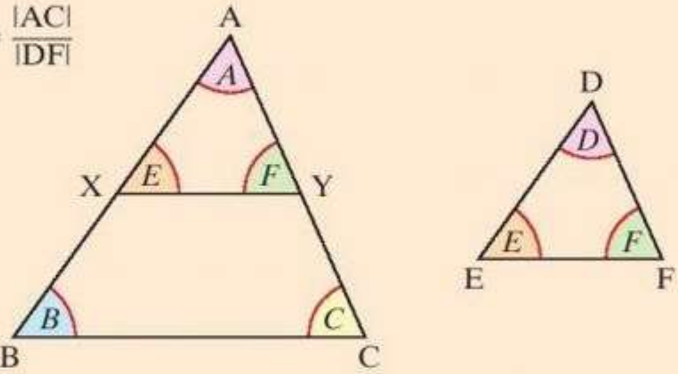


15. Draw separate diagrams of the triangles ABD and ACD. Mark in equal angles and explain why the two triangles are similar.
Hence find $|BD|$ and $|AB|$.



***Theorem 13** If two triangles ABC and DEF are similar, then their sides are proportional in order:

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$$



Given: The triangles ABC and DEF, in which
$|\angle A| = |\angle D|$, $|\angle B| = |\angle E|$ and $|\angle C| = |\angle F|$.

To Prove: $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$.

Construction: Mark the point X on [AB] such that $|AX| = |DE|$.
Mark the point Y on [AC] such that $|AY| = |DF|$.
Join XY.

Proof: The triangles AXY and DEF are congruent ...(SAS)

$\therefore\ |\angle AXY| = |\angle DEF|$ ...corresponding angles

$\therefore\ |\angle AXY| = |\angle ABC|$

$\therefore\ XY \| BC$

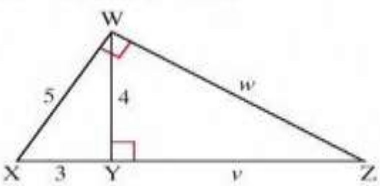
$\therefore\ \frac{|AB|}{|AX|} = \frac{|AC|}{|AY|}$ ...a line parallel to one side divides the other side in the same ratio

$\therefore\ \frac{|AB|}{|DE|} = \frac{|AC|}{|DF|}$

Similarly it can be proved that $\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|}$.

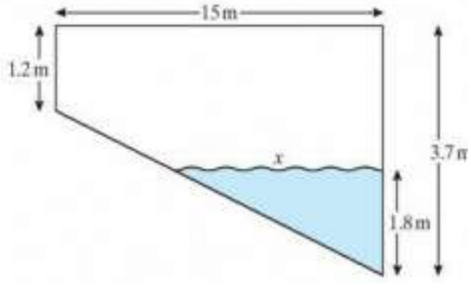
$\therefore\ \frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}$.

18. In the given figure, $|\angle WYZ| = |\angle XWZ| = 90°$.



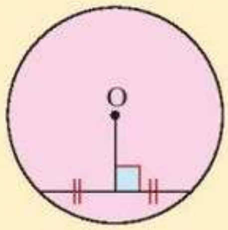
(i) Which triangle is similar to the triangle WXY?
(ii) Hence find the values of v and w.

21. The diagram shows the side-view of a swimming pool being filled with water.
Calculate the length of x.

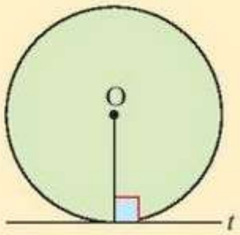


Summary of Key Points

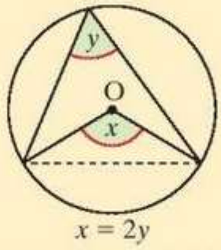
1. Circle properties



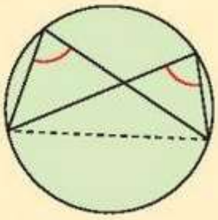
The perpendicular from the centre of a circle bisects the chord.



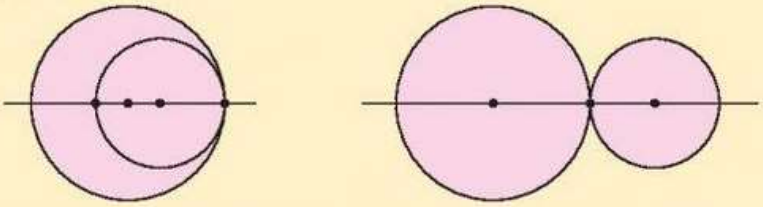
A tangent is perpendicular to the radius that goes to the point of contact.



The angle at the centre is twice the angle at the circumference.



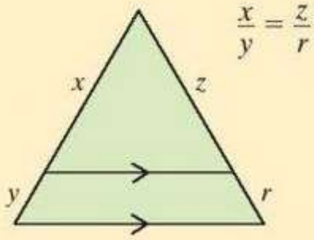
Angles in the same segment of a circle are equal.



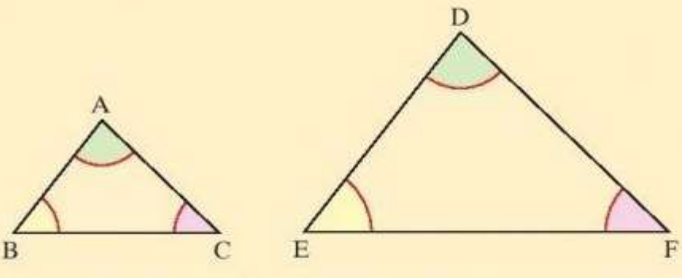
If two circles intersect at one point only, then their centres and the point of contact are collinear.

2. Triangles and parallelograms

1. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
2. The angle opposite the longer of two sides is greater than the angle opposite the shorter side.
3. For any triangle, base times height does not depend on the choice of base.
4. A diagonal of a parallelogram bisects the area.
5. The area of a parallelogram is the base multiplied by the perpendicular height.
6. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.
7. A line drawn parallel to one side of a triangle divides the other two sides in the same ratio.



8.



If two triangles ABC and DEF are similar, then their sides are proportional in order,

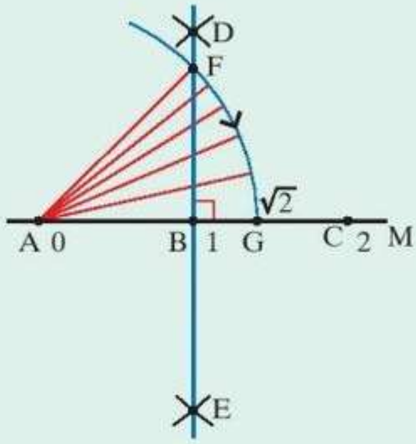
$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|}.$$

Constructing a line of length $\sqrt{2}$

Although $\sqrt{2} = 1.414214...$ is a non-terminating decimal, it is possible to construct a line of length $\sqrt{2}$ on the number line as the following example shows.

Example 2

Using a compass and straightedge only, construct a line segment of length $\sqrt{2}$ and hence mark $\sqrt{2}$ on the number line.



(i) Using a straightedge, draw a line segment [AM].

(ii) Starting at A, mark equal spaces 0, 1, 2... (A, B, C) using a compass.

(iii) Using a compass, construct the perpendicular bisector of [AC], that is, draw a perpendicular line through B.

(iv) Join D and E.

(v) Mark the point F on [DE] so that |AB| = |BF|.

(vi) Using A as the centre and |AF| as radius, draw an arc FG to the number line.

(vii) Mark G on the number line, $\sqrt{2}$.

Proof: Consider the triangle ABF:

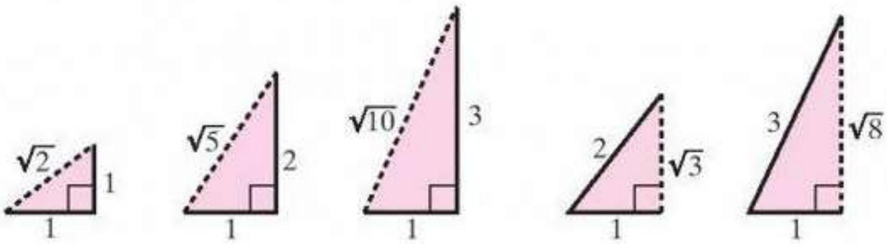
$|AB| = 1, |BF| = 1, ABF = 90°$

Using Pythagoras' theorem: $|AF|^2 = |AB|^2 + |FB|^2$

$\therefore\ |AF|^2 = 1^2 + 1^2 = 2$

$\therefore\ |AF| = \sqrt{2} \Rightarrow |AG| = \sqrt{2}$

Note: Using similar constructions, other irrational numbers can be plotted on the number line.



Example 3

Construct a line segment of length $\sqrt{3}$ on the number line.

(i) Mark a point A on a straight line AB.

(ii) Using a compass, mark equal spaces |AJ| and |JC| (each 1 unit) along AB.

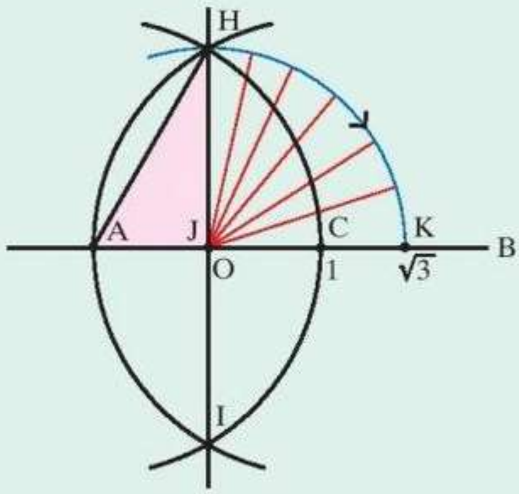
(iii) Using A as centre and |AC| as radius, draw an arc.

(iv) Using C as centre and |CA| as radius, draw an arc.

(v) Join the points of intersection of the arcs HI.

(vi) From our geometry theorems, we know that HI is perpendicular to AB and bisects [AC] at J.

Consider the triangle AJH.



$|AJ| = 1, |AH| = 2, \angle AJH = 90^\circ$

$\therefore \quad |AH|^2 = |AJ|^2 + |JH|^2$ ...using Pythagroas's theorem

$\therefore \quad 2^2 = 1^2 + |JH|^2$

$\therefore \quad 4 = 1 + |JH|^2$

$\therefore \quad 3 = |JH|^2 = >JH = \sqrt{3}$

Using J as centre and |JH| as radius, draw an arc HK intersecting the horizontal line at K. $\quad |JK| = \sqrt{3}$

Proof that $\sqrt{2}$ is irrational :

In a proof by contradiction, we show that if we claim some statement to be true, then a logical contradiction occurs which proves that the original statement must be false.

Example 2

Prove that $\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ is a rational number.

$\therefore$ $\sqrt{2}$ can be written as $\frac{a}{b} : a, b \in Z$ and $b \neq 0$, and a and b have no common factor.

$\therefore$ $2 = \frac{a^2}{b^2}$ ... squaring both sides

$\therefore$ $a^2 = 2b^2$

$\therefore$ a^2 is even since it is equal to $2 \times (b^2)$ .... any integer multiplied by 2 is even

$\therefore$ a is even

$\therefore$ a can be written as $a = 2c$ ... since a is divisible by 2

$\therefore$ $a^2 = 4c^2 = 2b^2$

$\therefore$ $2c^2 = b^2$

$\therefore$ b is also even

$\therefore$ a and b have a common factor of 2 which contradicts our assumption.

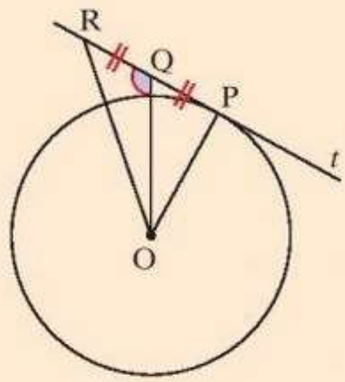
$\therefore$ $\sqrt{2}$ is not a rational number.

$\therefore$ $\sqrt{2}$ is an irrational number, i.e. it cannot be written as $\frac{a}{b}$ with no common factors.

2. Prove by contradiction that if a is a rational number and b an irrational number, then $a + b$ is an irrational number.

3. Prove by contradiction that there are no positive integer solutions to $x^2 - y^2 = 10$. (Hint: Assume that x and y are both positive integers and use the difference of two squares.)

Example of Theorem that uses Proof by contradiction:



Theorem 20 A tangent is perpendicular to the radius that goes to the point of contact.

Given: A tangent t to a circle of centre O. P is the point of contact of the tangent and circle and [OP] is the radius to the point of contact.

To prove: $OP \perp t$

Construction: Let the perpendicular to the tangent from the centre O meet it at Q. Pick another point R on t such that $|PQ| = |QR|$. Join OQ and OR.

Proof: In the triangles OPQ and OQR,

$|OQ| = |OQ|$ ...common side

$|PQ| = |QR|$ ...given

$|\angle OQP| = |\angle OQR|$ ...both 90°

$\therefore$ the triangles OPQ amd OQR are congruent

$\therefore$ $|OR| = |OP|$ ...both hypotenuses

So R is a second point where t meets the circle.
This contradicts the given fact that t is a tangent.
Thus t must be perpendicular to [OP], i.e., $OP \perp t$.

Section 3.9 De Moivre's theorem

In the previous section, we saw that

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_1 + i\sin\theta_1) = \cos(\theta_1+\theta_1) + i\sin(\theta_1+\theta_1)$$
$$= \cos 2\theta_1 + i\sin 2\theta_1$$

That is $\quad (\cos\theta_1 + i\sin\theta_1)^2 = \cos 2\theta_1 + i\sin 2\theta_1$

Also, $\quad (\cos\theta_1 + i\sin\theta_1)^3 = \cos 3\theta_1 + i\sin 3\theta_1$

The general case of this result is known as **de Moivre's theorem**.

De Moivre's theorem

> $(\cos\theta + i\sin\theta)^n \equiv \cos n\theta + i\sin n\theta$, for all real values of n.

Proof of de Moivre's Theorem by Induction

A: When n is a positive integer, prove $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

(i) Let $n = 1 \Rightarrow (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta \ldots$ which is true

(ii) Assume that $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$

(iii) Prove true for $n = k+1$, i.e.,

$$(\cos\theta + i\sin\theta)^{k+1} = [\cos(k+1)\theta + i\sin(k+1)\theta]$$

Proof: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\ldots$ assumed

$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta$

$= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

Therefore, if it is true for $n = k$, it is true for $n = k+1$.

But it is true for $n = 1$.

Thus, it is true for $n = 1 + 1 = 2$.

Therefore, the theorem is true for $n = 1, 2, 3, \ldots$. i.e. for all positive integers.

B: When n is a negative integer,

let $n = -p$ where p is a positive integer.

We now prove that $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

$$\Rightarrow (\cos\theta + i\sin\theta)^{-p} = \frac{1}{(\cos\theta + i\sin\theta)^p}$$

$$= \frac{1}{\cos p\theta + i\sin p\theta} \quad \text{... using de Moivre's theorem}$$

$$= \frac{1}{\cos p\theta + i\sin p\theta}\cdot\frac{\cos p\theta - i\sin p\theta}{\cos p\theta - i\sin p\theta}$$

$$= \frac{\cos p\theta - i\sin p\theta}{\cos^2 p\theta + \sin^2 p\theta}$$

$$= \cos p\theta - i\sin p\theta$$

> $\cos(-\theta) = \cos\theta$
> $\sin(-\theta) = -\sin\theta$

But $p = -n$.

$\therefore\ (\cos\theta + i\sin\theta)^n = \cos(-n\theta) - \sin(-n\theta)$

$= \cos n\theta + i\sin n\theta$

$\therefore\ (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all negative integers.

Note: For $n = 0$, $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

becomes $(\cos\theta + i\sin\theta)^0 = (\cos 0 + i\sin 0)$

$1 = 1$

Therefore, $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ for all integer values.

> If $z = r(\cos\theta + i\sin\theta)$, then using de Moivre's Theorem:
> $z^n = [r(\cos\theta + i\sin\theta)]^n$
> $= r^n(\cos n\theta + i\sin n\theta)$ for all $n \in Z$.

Section 4.5 Geometric series

When the terms of a geometric sequence are added, a **geometric series** is created.

For example, $2 + 6 + 18 + 54 + \ldots\ldots$ is a geometric series.

To find the sum of a geometric series, we use the following procedure:

$$S_n = a + ar + ar^2 + \ldots\ldots ar^{n-3} + ar^{n-2} + ar^{n-1}$$

$$\Rightarrow \quad r.S_n = \quad ar + ar^2 + \ldots\ldots ar^{n-3} + ar^{n-2} + ar^{n-1} + ar^n$$

... multiplying each term by r

Subtracting: $S_n - rS_n = a - ar^n$

$$\therefore \quad S_n(1 - r) = a(1 - r^n)$$

$$\therefore \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

> The sum to n terms of a geometric sequence,
>
> $S_n = \dfrac{a(1 - r^n)}{1 - r}$, where a is the first term and r is the common ratio.

Now consider the formula for the sum to n terms of a geometric series if $|r| < 1$.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

When $|r| < 1$, r^n will approximate to zero for large values of n, i.e. $r^n \to 0$ as $n \to \infty$.

Thus, $S_n = \dfrac{a(1 - r^n)}{1 - r}$ becomes $S_n = \dfrac{a(1 - 0)}{1 - r}$

$$\therefore \quad S_n = \frac{a}{1 - r} \quad \text{as} \quad n \to \infty.$$

$$\text{or} \quad \lim_{n \to \infty} S_n = \frac{a}{1 - r}$$

> For a geometric series with $|r| < 1$,
>
> $\lim\limits_{n \to \infty} S_n = \dfrac{a}{1 - r}$.

Example 4

Write the recurring decimal $0.\dot{2}\dot{3}$ as a fraction in the form $\frac{a}{b}$, $a, b, \in N$.

$$0.\dot{2}\dot{3} = 0.232323\ldots\ldots = 0.23 + 0.0023 + 0.000023 + \ldots\ldots$$

$$= \frac{23}{100} + \frac{23}{10000} + \frac{23}{1000000} + \ldots\ldots$$

$$\Rightarrow \quad a = \frac{23}{100} \quad \text{and} \quad r = \frac{23}{10000} \div \frac{23}{100} = \frac{1}{100}$$

$$\Rightarrow \quad \lim_{n \to \infty} S_n = \frac{a}{1 - r} = \frac{\frac{23}{100}}{1 - \frac{1}{100}} = \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$$

[Note: $\lim\limits_{n \to \infty} S_n$ is often written as S_∞. Thus, $S_\infty = \frac{23}{99}$]

Hence express each as a decimal in the form $\frac{a}{b}$, $a, b \in N$.

(i) $0.\dot{7}$ (ii) $0.\dot{3}\dot{5}$ (iii) $0.2\dot{3}$ (iv) $0.\dot{3}7\dot{0}$ (v) $0.1\dot{6}\dot{2}$ (vi) $0.\dot{3}2\dot{1}$

Proof by Induction:

Example 1

Prove that **for all values of n**, $1 + 2 + 3 + 4 + \dots. n = \frac{n}{2}(n + 1)$.

Proof:

(i) Prove the statement true for $n = 1$.

$\Rightarrow 1 = \frac{1}{2}(1 + 1) = \frac{1}{2}(2) = 1$, which is true.

(ii) Assume true for $n = k$.

$\Rightarrow 1 + 2 + 3 + 4 + \dots. k = \frac{k}{2}(k + 1)$.

(iii) Based on this assumption, we must show that the statement is true for $n = k + 1$.

$1 + 2 + 3 + 4 + \dots. k + \mathbf{(k + 1)} = \frac{k}{2}(k + 1) + \mathbf{(k + 1)}$ ... adding $(k + 1)$ to both sides.

$= (k + 1)\left(\frac{k}{2} + 1\right)$ ... factorising $(k + 1)$ from RHS.

$= (k + 1)\left(\frac{k + 2}{2}\right)$ ... getting a common denominator.

$= \left(\frac{k + 1}{2}\right)(k + 2)$ ... re-arranging the denominator.

$= \left(\frac{k + 1}{2}\right)[(k + 1) + 1]$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.
And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, ... etc.

(v) Therefore, it is true for all values of n.

Example 3

Prove by induction that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots. \frac{1}{n(n+1)} = \frac{n}{n+1}, n \in N.$

Proof:

(i) Prove the statement true for $n = 1$.

$$\frac{1}{1(1+1)} = \frac{1}{1+1} = \frac{1}{2} \quad \text{... which is true.}$$

(ii) Assume true for $n = k$.

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots. \frac{1}{k(k+1)} = \frac{k}{k+1}, k \in N.$$

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots. \frac{1}{k(k+1)} + \mathbf{\frac{1}{(k+1)(k+2)}} = \frac{k}{k+1} + \mathbf{\frac{1}{(k+1)(k+2)}}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)} = \frac{(k+1)}{(k+1)+1}$$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$. And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, ... etc.

(v) Therefore, it is true for all values of n.

3. $1.2 + 2.3 + 3.4 + 4.5 + \ldots. n(n+1) = \sum_{n=1}^{n} n(n+1) = \frac{n}{3}(n+1)(n+2).$

4. $\frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \frac{1}{5.6} + \ldots. \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}.$

Example 4

Prove that for all $n \in N$, 3 is a factor of $4^n - 1$.

Proof:

(i) Prove the statement true for $n = 1$.

3 is a factor of $4^1 - 1 = 3$ ... true.

(ii) Assume true for $n = k$.

$\Rightarrow$ 3 is a factor of $4^k - 1, k \in N$.

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

Is 3 a factor of $4^{k+1} - 1$?

$$= 4^k.4^1 - 1$$
$$= 4^k.(3 + 1) - 1$$
$$= 3.4^k + 1.4^k - 1$$
$$= 3.4^k + (4^k - 1).$$

Since 3.4^k is divisible by 3 and $(4^k - 1)$ is assumed divisible by 3,

$\therefore$ $3.4^k + (4^k - 1)$ is divisible by 3.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, ... etc.

(v) Therefore, it is true for all values of n.

1. $6^n - 1$ is divisible by 5 for $n \in N$.

3. $9^n - 5^n$ is divisible by 4 for $n \in N$.

Example 5

Prove by induction that $8^n - 7n + 6$ is divisible by 7 for all $n \in N$.

Proof:

(i) Prove the statement true for $n = 1$.

7 is a factor of $8^1 - 7.1 + 6 = 7$ ... which is true.

(ii) Assume true for $n = k$.

$\Rightarrow$ 7 is a factor of $8^k - 7k + 6, k \in N$.

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

Is 7 a factor of $8^{k+1} - 7(k + 1) + 6$?

$$= 8^k.8^1 - 7k - 7 + 6$$
$$= 8^k.(7 + 1) - 7k - 7 + 6$$
$$= 7.8^k + 1.8^k - 7k - 7 + 6$$
$$= 7.8^k + (8^k - 7k + 6) - 7$$

Since 7.8^k is divisible by 7, $(8^k - 7k + 6)$ is assumed divisible by 7 and -7 is divisible by 7,

$\therefore$ $8^{k+1} - 7(k + 1) + 6$ is divisible by 7.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$.

And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, ... etc.

(v) Therefore, it is true for all values of n.

3. $9^n - 5^n$ is divisible by 4 for $n \in N$.

9. $7^n + 4^n + 1$ is divisible by 6 for $n \in N$.

10. $n(n + 1)(2n + 1)$ is divisible by 3 for $n \in N$.

Inequality proofs

When dealing with inequalities we noted two important deductions, namely

(i) If $a > b$, then $a - b > 0$

(ii) (any real number $)^2 > 0$.

Example 7

Prove by induction that $2^n > n^2$ for $n \geqslant 5, n \in N$.

Proof:

(i) Prove the statement true for $n = 5$.

$2^5 > 5^2$

$32 > 25$ ... which is true.

(ii) Assume true for $n = k, k \geqslant 5$.

$\Rightarrow 2^k > k^2, k \in N$ and $k \geqslant 5$.

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

Is $2^{k+1} > (k + 1)^2$?

Since $2^k > k^2$ (assumed)

$\therefore \quad 2^k . 2 > 2k^2$

$\therefore \quad 2^{k+1} > 2k^2$

$\therefore$ we need to prove that $2k^2 > (k + 1)^2$.

$2k^2 > k^2 + 2k + 1$

$k^2 - 2k - 1 > 0$

$k^2 - 2k \mathbf{+ 1 - 1} - 1 > 0$ ... completing the square by adding and subtracting half the coefficient of k squared.

$k^2 - 2k + 1 - 2 > 0$

$(k - 1)^2 - 2 > 0$ which is true for $k \geqslant 5$.

$\therefore \quad 2^{k+1} > 2k^2 > (k + 1)^2$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 5$, it now must be true for $n = 5 + 1 = 6$. And if it is true for $n = 6$, it is true for $n = 6 + 1 = 7$, ... etc.

(v) Therefore, it is true for all values of $n \geqslant 5, n \in N$.

Example 8

Prove by induction that $n! > 2^n, n \geqslant 4, n \in N$.

Proof:

(i) Prove the statement true for $n = 4$.

$4! > 2^4$

$24 > 16$ ... which is true.

(ii) Assume true for $n = k, k \geqslant 4$.

$\Rightarrow k! > 2^k, k \in N$ and $k \geqslant 4$.

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

Is $(k + 1)! > 2^{k+1}$?

Is $(k + 1)k! > 2^k.2$? ... $(k + 1)k! = (k + 1)!$

Since $k! > 2^k$ (assumed),

$\therefore \quad (k + 1)k! > (k + 1)2^k$

$\therefore$ we need to prove that $(k + 1)2^k > 2^k.2$

$(k + 1)2^k > 2.2^k$ which is obviously true if $k > 1$.

$\therefore \quad (k + 1)! > 2^{k+1}$

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 4$, it now must be true for $n = 4 + 1 = 5$. And if it is true for $n = 5$, it is true for $n = 5 + 1 = 6$, ... etc.

(v) Therefore, it is true for all values of $n \geqslant 4, n \in N$.

Example 9

Prove that $(1 + x)^n \geqslant 1 + nx$ for $n \geqslant 1, n \in N, x \in R$.

Proof:

(i) Prove the statement true for $n = 1$.

$\Rightarrow (1 + x)^1 \geqslant 1 + 1.x$ ... which is true.

(ii) Assume true for $n = k, k \geqslant 1$.

$(1 + x)^k \geqslant 1 + kx,\ k \in N$ and $k \geqslant 1$.

(iii) Based on this assumption, we must now show that the statement is true for $n = k + 1$.

Is $(1 + x)^{k+1} \geqslant 1 + (k + 1)x$?

Is $(1 + x)^k(1 + x) \geqslant 1 + kx + x$?

Since $(1 + x)^k \geqslant 1 + kx$ ... (assumed).

$\therefore \quad (1 + x)^k(1 + x) \geqslant (1 + kx)(1 + x)$

$\therefore$ we need to prove that $(1 + kx)(1 + x) \geqslant 1 + kx + x$

$1 + x + kx + kx^2 \geqslant 1 + kx + x.$

$\Rightarrow kx^2 \geqslant 0$ which is true for $k \geqslant 1$ and $x \in R$.

$\therefore$ It is true for $n = k + 1$.

(iv) But since it is true for $n = 1$, it now must be true for $n = 1 + 1 = 2$. And if it is true for $n = 2$, it is true for $n = 2 + 1 = 3$, ... etc.

(v) Therefore, it is true for all values of $n \geqslant 1, n \in N$.