Proof That $\sqrt{3}$ Is Irrational

To prove: $\sqrt{3}$ is irrational.

The proof of this result is another example of proof by contradiction.

Proof: Assume that $\sqrt{3}$ is rational and can therefore be written in the form $\frac{a}{b}$, $a, b \in Z, b \neq 0$.

Also, assume that the fraction $\frac{a}{b}$ is written in simplest terms, i.e. HCF$(a, b) = 1$.

$$\sqrt{3} = \frac{a}{b},$$

$$\Rightarrow 3 = \frac{a^2}{b^2} \text{ (squaring both sides)}$$

$$\therefore a^2 = 3b^2 \qquad (*)$$

As b^2 is an integer, a^2 has to be a multiple of 3, which means that 3 divides a^2.

If 3 divides a^2, then 3 divides a. (Worked Example 1.3)

$\therefore a = 3k$, for some integer k. Substituting $3k$ for a in (*) gives,

$$(3k)^2 = 3b^2$$

$$9k^2 = 3b^2$$

$$\Rightarrow b^2 = 3k^2$$

As k^2 is an integer, b^2 has to be a multiple of 3, which means that 3 divides b^2.

Therefore, 3 divides b. If 3 divides a and 3 divides b, then this contradicts the assumption that HCF$(a, b) = 1$. This completes the proof.

Prove that $\sqrt{2}$ is irrational.

Assume $\sqrt{2} = \frac{a}{b}$; $a, b \in Z, b \neq 0$

$\Rightarrow 2 = \frac{a^2}{b^2}$

$\Rightarrow a^2 = 2b^2$

$\Rightarrow 2 | a^2$

$\Rightarrow 2 | a$

$\Rightarrow a = 2m, m \in Z$

$a^2 = 2b^2$

$\Rightarrow (2m)^2 = 2b^2$

$\Rightarrow 4m^2 = 2b^2$

$\Rightarrow b^2 = 2m^2$

$\Rightarrow 2 | b^2$

$\Rightarrow 2 | b$

$\Rightarrow b = 2n, n \in Z$

$\therefore \frac{a}{b} = \frac{2m}{2n} = \frac{m}{n}$

$\therefore \frac{a}{b}$ can be simplified to $\frac{m}{n}$

By similar argument, $\frac{m}{n}$ can be simplified ad infinction. This is absurd.

$\therefore$ initial assumption that $\sqrt{2} = \frac{a}{b}$ is incorrrect.

$\therefore \sqrt{2}$ is irrational.