<u>Section 7.11</u> Problem Solving

Exponential and log functions are used to model many real life problems.

- Loudness of sound
- Acidity of a solution
- Intensity of earthquakes

Example 1

The acidity of a substance is determined by the ion concentration formula $pH = -\log[H^+]$, where a pH of 7 is defined as neutral, <7 acidic, >7 alkaline. Determine the acidity of each of the following substances.

(a) Apple juice with a $[H^+]$ ion concentration of 0.0003.
(b) Ammonia with a $[H^+]$ ion concentration of 1.3×10^{-9}.

<u>example 1</u>

$pH = -\log[H^+]$

ph < 7 acidic
ph > 7 alkaline

(a) Acidity of Apple Juice with $[H^+] = 0.0003$.

$pH = -\log[0.0003] = 3.52$ (cal)

$\Rightarrow$ acidic

(b) Ammonia with $H^+ = 1.3 \times 10^{-9}$

$pH = -\log(1.3 \times 10^{-9}) = 8.88 \Rightarrow$ alkaline.

Example 2

The loudness of a sound of intensity I is given by the formula $\text{dB} = 10\log\left(\frac{I}{I_o}\right)$, where dB is measured in decibels and I_o is the threshold intensity of hearing ($I_o = 1 \times 10^{-12}\,\text{Wm}^{-2}$).

(a) Find the loudness (in decibels) of a sound at the threshold of hearing.

(b) Given that prolonged exposure to sounds over 85 decibels can cause hearing damage, and that a gunshot from a .22 rifle has an intensity of $\boxed{I = 2.5 \times 10^{13} I_o}$, should you wear ear protection when firing this gun?

$$dB = 10 \log\left(\frac{I}{I_o}\right) \qquad I_o = 1 \times 10^{-12}$$

(a) find dB at threshold of hearing.

$$db = 10 \log \frac{1 \times 10^{-12}}{1 \times 10^{-12}} = 10 \log 1 = 0$$

(b) Damage @ >85 db   Is $I = 2.5 \times 10^{13} I_o$ dangerous to hearing?

$$db = 10 \log \frac{(2.5 \times 10^{13})(1 \times 10^{-12})}{(1 \times 10^{-12})} = 10 \log (2.5 \times 10^{13}) = 134$$

yes.

Compound Interest

$A = P(1 + i)^t$ tables pg 30

To calculate time → use logs

$A = P(1 + i)^t$ ⟶ Find t.

$(\div P)$ $\frac{A}{P} = (1+i)^t$

$(\times Ln)$ $Ln\left(\frac{A}{P}\right) = Ln(1+i)^t$

$Ln\left(\frac{A}{P}\right) = t\, Ln(1+i)$

$\frac{Ln\left(\frac{A}{P}\right)}{Ln(1+i)} = t.$

Use calculator to find t.

Example 3

How long would it take €5000 to increase in value to €6000, if invested in a credit union at a yearly compound interest rate of 2%?

Ex 3 5000 → 6000 at 2% CI find t.

$$A = P(1+i)^t$$

$$6000 = 5000(1.02)^t$$

$$\frac{6}{5} = (1.02)^t$$

$$\ln \frac{6}{5} = \ln (1.02)^t$$

$$\ln 6/5 = t \ln (1.02)$$

$$\frac{\ln\left(\frac{6}{5}\right)}{\ln(1.02)} = t$$

$$9.21_{\text{years}} = t$$

~~.21 × 12~~

$.21 \times 365$

$= 77$ days

9yrs 77 days = t

Depreciation

$F = P(1 - i)^t$ tables pg 30

Example 4

The population of red squirrels in a given region was estimated to be 5000 at the start of 2003. Assuming a rate of decrease of 5% per year, estimate the size of the population in 2013.

Eg: Squirrels in 2003 = 5000

↓ 5% ? in 2013

$$F = 5000(1 - 0.05)^{10}$$

$$= 5000(0.95)^{10}$$

$= 2994$ squirrels

Doubling Time

Is the time required for a quantity to double in size.

The quantity at a given time, t, is

$y = Ae^{bt}$

A = start value,   b = growth constant,

If doubles $\rightarrow 2A = Ae^{bt}$

Divide by A $\rightarrow 2 = e^{bt}$

$$\ln 2 = \ln e^{bt}$$

$$\ln 2 = bt \ln e$$

$$\frac{\ln 2}{b \ln e} = t$$

But ln e = 1

$$\boxed{\frac{\text{Ln } 2}{b} = t}$$

Example 5

A certain type of bacteria is growing exponentially, where $y = A e^{bt}$ is the number of bacteria present after t (hours) and b is the growth constant. Under certain conditions, the bacteria doubles in population every 6.5 hours. If at the start of the experiment under these conditions there are 100 bacteria present, find (i) the growth constant b (ii) how many bacteria will be present after 2 days.

Ex 7·11

Q1 €5000 0·6% per Month C.I.

(i) (a) $F = P(1+i)^t$

$F = 5000(1+0\cdot006)^1$

$F = 5000(1\cdot006)^1$

$= €5030$

(b) $F = 5000(1\cdot006)^2$

$= €5060\cdot18$

(c) $F = 5000(1\cdot006)^3$

$= €5090\cdot54$

(ii) after t Months $\Rightarrow F = 5000(1\cdot006)^t$

(iii) Double Money ⇒ €10,000

$$10{,}000 = 5000(1.006)^t$$
$$2 = (1.006)^t$$
$$\ln 2 = t \cdot \ln 1.006$$
$$\frac{\ln 2}{\ln 1.006} = t$$
$$115.87 = t$$

⇒ Min time 116 months = 9yrs 219 days.

Q2 100 at start. - after 6hrs, 450

$y = Ae^{bt}$ find b

first find A: $100 = Ae^{b(0)}$ $(e^0 = 1)$

$100 = A$

$\Rightarrow y = 100e^{bt}$ $t = 6hrs$

Next find b

$450 = 100e^{b(6)}$

$4 \cdot 5 = e^{6b}$

$\ln 4 \cdot 5 = 6b \ln e$ $(\ln e = 1)$

$\ln 4 \cdot 5 = 6b$

$\frac{\ln 4 \cdot 5}{6} = b$

$0 \cdot 25 = b$

Q3 $45^\circ C$ $T = 15 + 30 \times 10^{-0.02t}$

(i) Initial temp $\Rightarrow t = 0$

$$T = 15 + \left(30 \times 10^{-0.02(0)}\right)$$
$$15 + \left(30 \times 10^{0}\right)$$
$$15 + (30 \times 1)$$
$$= 45^\circ \quad \text{True}$$

(ii) Cooled to 35^0

$$35 = 15 + 30 \times 10^{-0.02t}$$
$$20 = 30 \times 10^{-0.02t}$$
$$2/3 = 10^{-0.02t}$$
$$\log 2/3 = -0.02t \log 10 \qquad \left(\overset{\text{note}}{\log 10 = 1}\right)$$
$$\frac{\log 2/3}{-0.02} = t \qquad t = 8.8 \text{ mins.}$$

(iii) find room temp.

as t gets bigger $30 \times 10^{-0.02t}$ gets smaller until it equals zero.

$$\Rightarrow T = 15 + \left(30 \times 10^{-0.02t}\right)$$
$$= 15 + 0$$
$$= 15^\circ \quad \text{room temp.}$$

Q4 $\quad L = 10 \log_{10}\left(\frac{I}{I_0}\right) = 10 \log_{10}\left(\frac{I}{1 \times 10^{-12}}\right)$

(i) $L = 100$

$$100 = 10 \log_{10}\left(\frac{I}{1 \times 10^{-12}}\right)$$

$$100 = 10\left[\log_{10} I - \log_{10}\left(1 \times 10^{-12}\right)\right]$$

Q4 $L = 10\log_{10}\left(\frac{I}{I_0}\right) = 10\log_{10}\left(\frac{I}{1\times10^{-12}}\right)$

(i) $L = 100$

$$100 = 10\log_{10}\left(\frac{I}{1\times10^{-12}}\right)$$

$$100 = 10\left[\log_{10} I - \log_{10}\left(1\times10^{-12}\right)\right]$$

$$10 = \log_{10} I - (-12)$$

$$10 = \log_{10} I + 12$$

$$-2 = \log_{10} I$$

$$10^{-2} = I$$

$$0.01 = I$$

$L = 110$

$$110 = 10\log_{10}\left(\frac{I}{1\times10^{-12}}\right)$$

$$11 = \log_{10} I - \log_{10}\left(1\times10^{-12}\right)$$

$$11 = \log_{10} I + 12$$

$$-1 = \log_{10} I$$

$$10^{-1} = I$$

$$0.1 = I$$

$\Rightarrow$ Range is between $0.01\,Wm^{-2}$ and $0.1\,Wm^{-2}$

(ii) Pain $= 10\ Wm^{-2}$ $\Rightarrow I = 10$

$$L = 10\log_{10}\left(\frac{10}{1\times10^{-12}}\right)$$

$$L = 10\log_{10} 10^{13}$$

$$L = 10(13)(\log_{10}10)$$
$$= 10(13)(1)$$
$$= 130\ dB.$$

Q5

$A = 10^{m}$ $E = 10^{1.5m+4.8}$

find a and b $E = 10^{a} A^{b}$

$\Rightarrow E = 10^{a}\cdot 10^{mb}$

$$E = \left(10^{1.5m}\right)\left(10^{4.8}\right)$$
$$= \left(10^{1.5}\right)^{m}\left(10^{4.8}\right)$$
$$= \left(10^{m}\right)^{1.5}\left(10\right)^{4.8}$$
$$A^{1.5}\ 10^{4.8}$$

$\Rightarrow a = 4.8$ and $b = 1.5$.

Q5

$A = 10^{m}$ $\qquad E = 10^{1.5m + 4.8}$

find a and b $\qquad E = 10^{a} A^{b}$

$\Rightarrow E = 10^{a} . 10^{mb}$

$$E = \left(10^{1.5m}\right)\left(10^{4.8}\right)$$
$$= \left(10^{1.5}\right)^{m}\left(10^{4.8}\right)$$
$$= \left(10^{m}\right)^{1.5}\left(10\right)^{4.8}$$
$$A^{1.5} \; 10^{4.8}$$

$\Rightarrow a = 4.8$ and $b = 1.5$.

Q6 €100 in 2000 rising at 4.5%

(i) $F = P(1+i)^{t}$

$F = 100(1.045)^{t}$

(ii) 2010 $\Rightarrow t = 10$ $\qquad \Rightarrow F = 100(1.045)^{10} = €155.30$.

(iii) 2005 5 yr previous $\Rightarrow t = -5$.

$F = 100(1.045)^{-5} = €80.25$

Q7 $W = 0{\cdot}6 \times 1{\cdot}15^t$

(i) at birth $\Rightarrow t = 0$ $W = 0{\cdot}6 \times 1{\cdot}15^0$

$= 0{\cdot}6 \times 1 = 0{\cdot}6\,kg$

(ii) $1{\cdot}15 \Rightarrow$ 15%

(iii) Double weight $\Rightarrow 2(0{\cdot}6) = 1{\cdot}2\,kg$.

$$1{\cdot}2kg = 0{\cdot}6 \times 1{\cdot}15^t$$
$$2 = 1{\cdot}15^t$$
$$\ln 2 = t \ln 1{\cdot}15$$
$$\frac{\ln 2}{\ln 1{\cdot}15} = t$$
$$4{\cdot}959 = t$$

$\Rightarrow$ approx 5 months

approx 5 months

Q8

$$M = M_0 e^{-kt}$$

(i) $10 = M_0 e^{-k(0)}$ $M = 10$ when $t = 0$.

$10 = M_0 e^0$

$10 = M_0 (1)$

$\underline{10 = M_0}$

$5 = 10 e^{-k(140)}$ $m = 5$ when $t = 140$.

$0.5 = e^{-140k}$

$\ln 0.5 = -140k \ln e$

$\ln 0.5 = -140k(1)$

$\dfrac{\ln 0.5}{-140} = k$

$4.95 \times 10^{-3} = k$

$0.00495 = k$

(ii) Mass after 70 days

$$M = 10e^{-0.00495t}.$$

$$M = 10e^{-0.0045(70)}$$

$$M = 10e^{-0.3465}$$

$$= 10(0.7072)$$

$$= 7.072 = 7g.$$

(iii) 2g. left. $\Rightarrow M = 2g$

$$2 = 10e^{-0.00495t}$$

$$0.2 = e^{-0.00495t}$$

$$\ln 0.2 = -0.00495t(\ln e) \qquad [\ln e = 1]$$

$$\frac{\ln 0.2}{-0.00495} = t$$

$$325 = t$$

325 days.