Section 7.9 Logarithmic Functions

$x = 2^5$ then $x = 32$

$200 = 2^x$ to find x we need to use logs → $\log_2 200 = x$

$32 = 2^5$ using log notation $\log_2 32 = 5$

This is read "the log of 32 to the base 2 is 5"

$32 = 2^5$

$\text{Log}_2 32 = 5$

To make the power, drop the base number to in-front of the log and the power drops down.

Write the following in log form

$10^2 = 100$ $\quad \text{Log}_{10} 100 = 2$

$5^4 = 625$ $\quad \log_5 625 = 4$

$a^x = y$ $\quad \text{Log}_a y = x$

The logarithm of a number is the power to which the base number must be raised to get that number.

$10^2 = 100$ $\qquad$ $\text{Log}_{10}100 = 2$

$5^4 = 625$ $\qquad$ $\text{Log}_5 625 = 4$

$a^x = y$ $\qquad$ $\text{Log}_a y = x$

Reversed: from logs to indices

$\text{Log}_5 25 = 2$ $\qquad$ $5^2 = 25$

The base goes over and pushes the number up to become the power.

Try the following:

$\text{Log}_3 27 = 3$ $\qquad$ $3^3 = 27$

$\text{Log}_2 16 = 4$ $\qquad$ $2^4 = 16$

$\log_p R = s$ $\qquad$ $p^s = R$

Example 1

Evaluate (i) $\log_9 27$ (ii) $\log_{\frac{1}{3}} 9$ (iii) $\log_{\sqrt{2}} 8$.

(i) $\log_9 27 = x$

$9^x = 27$

$3^{2x} = 3^3$

$2x = 3$

$x = 3/2$

$\log_{\frac{1}{3}} 9 = x$

$\frac{1}{3}^x = 9$

$3^{-x} = 3^2$

$-x = 2$

$x = -2$

$\log_{\sqrt{2}} 8 = x$

$\sqrt{2}^x = 8$

$2^{x/2} = 2^3$

$\frac{x}{2} = 3$

$x = 6$

The Laws of Logs

The Laws of Logarithms

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a x^n = n\log_a x$
4. $\log_a a = 1$
5. $\log_a 1 = 0$
6. $\log_a x = \dfrac{\log_b x}{\log_b a}$

Tables pg 21

Using your calculator, verify each of the following.

1. $\log_{10}4 + \log_{10}3 = \log_{10}12 = 1.0792$
2. $\log_{10}8 - \log_{10}6 = \log_{10}\left(\frac{8}{6}\right) = 0.1249$
3. $\log_{10}8^3 = 3\log_{10}8 = 2.7093$
4. $\log_{10}10 = 1$
5. $\log_{10}1 = 0$

The two most widely used bases on logs are

- Base 10 logs eg $\log_{10}1000$. These are used for calculations and are called common logs. To calculate correct to dec places/sig figs change all to the base 10 or convert to logs by multiplying both sides by log. (See Q8)
- Base e (2.718) eg $\log_e 1000$. These are used for dealing with naturally occurring events, earthquakes, growth of colonies, and are called natural logs.
  Note: $\log_e x = \ln x$.

Example 2

Without using a calculator, simplify the following number:

$$2\log_{10}3 + \log_{10}16 - 2\log_{10}\left(\frac{6}{5}\right)$$

$$\log_{10}3^2 + \log_{10}16 - 2\log_{10}\left(\frac{6}{5}\right)$$

$$\log_{10}9 + \log_{10}16 - \log_{10}\left(\frac{6}{5}\right)^2$$

$$\log_{10}(9\times16) - \log_{10}\left(\frac{36}{25}\right)$$

$$\log_{10}\frac{144}{36/25}$$

$$\log_{10}\left(144\times\frac{25}{36}\right) = \log_{10}100 = x$$

$$10^x = 100$$

$$x = 2.$$

Example 3

Without using a calculator, simplify the following number:

$$\log_2 128 + \log_3 45 - \log_3 5$$

$$\log_2 128 + \log_3\left(\frac{45}{5}\right)$$

$$\log_2 128 + \log_3 9$$

$$7 + 2$$

$$= 9$$

Example 4

Evaluate the following number correct to two significant figures:

$\log_8 11 - \log_6 4$

$$\frac{\log_{10} 11}{\log_{10} 8} - \frac{\log_{10} 4}{\log_{10} 6}$$

$$= \frac{1.041}{0.903} = 1.153 \quad = \frac{.602}{.778} = 0.774$$

$$1.153 - 0.774 = 0.38$$

Q'S 1 – 22

Exc 7.9

Q1 (i) $\log_2 4 = x$

$2^x = 4$

$2^x = 2^2$

$x = 2$

(ii) $\log_3 81 = x$

$3^x = 81$

$3^x = 3^4$

$x = 4$

(iii) $\log_{10} 1000 = x$

$10^x = 1000$

$10^x = 10^3$

$x = 3$

(iv) $\log_2 64 = x$

$2^x = 64$

$2^x = 2^6$

$x = 6$

Q2 (i) $\log_8 16 = x$

$8^x = 16$

$2^{3x} = 2^4$

$3x = 4$

$x = 4/3$

(ii) $\log_9 27 = x$

$9^x = 27$

$3^{2x} = 3^3$

$2x = 3$

$x = 3/2$

(iii) $\log_4 32 = x$

(iv) $\log_? 8 = x$

Q2 (i) $\log_8 16 = x$

$8^x = 16$

$2^{3x} = 2^4$

$3x = 4$

$x = 4/3$

(ii) $\log_9 27 = x$

$9^x = 27$

$3^{2x} = 3^3$

$2x = 3$

$x = 3/2$

(iii) $\log_{16} 32 = x$

$16^x = 32$

$2^{4x} = 2^5$

$4x = 5$

$x = 5/4$

(iv) $\log_{\frac{1}{2}} 8 = x$

$\left(\frac{1}{2}\right)^x = 8$

$2^{-x} = 2^3$

$x = -3$

(v) $\log_{\frac{1}{3}} 81 = x$

$\left(\frac{1}{3}\right)^x = 81$

$3^{-x} = 3^4$

$x = -4$

Q3 (i) $\log_{\frac{1}{3}} 27 = x$

$\left(\frac{1}{3}\right)^x = 27$

$3^{-x} = 3^3$

$x = -3$

(ii) $\log_{\sqrt{2}} 4 = x$

$\sqrt{2}^{x} = 4$

$2^{\frac{1}{2}x} = 2^2$

$x = 4$

(iii) $\log_8 x = 2$

$8^2 = x$

~~64=~~

$64 = x$

(iv) $\log_{64} x = \frac{1}{2}$

$64^{\frac{1}{2}} = x$

$\sqrt{64} = x$

$8 = x$

Q4 (i) $\log_2 x = -1$

$2^{-1} = x$

$\frac{1}{2} = x$

(ii) $\log_3 \sqrt{27} = x$

$3^x = \sqrt{27}$

$3^x = 3^{3/2}$

$x = 3/2$

$\frac{1}{2} = x$ $\qquad$ $3^x = 3^{3/2}$

$x = 3/2$

(iii) $\log_x 2 = 2$

$x^2 = 2$

$x = \sqrt{2}$

(iv) $\log_2(0\cdot 5) = x$

$2^x = 0\cdot 5$

$2^x = 1/2$

$2^x = 2^{-1}$

$x = -1$

Q5 (i) $\log_4 2 + \log_4 32$

$\Rightarrow \log_4(2 \times 32) = x$

$\Rightarrow \log_4 64 = x$

$4^x = 64$

$4^x = 4^3$

$x = 3$

(ii) $\log_6 9 + \log_6 8 - \log_6 2$

$\Rightarrow \log_6 \frac{(9 \times 8)}{2}$

$\Rightarrow \log_6 36 = x$

$6^x = 36$

$x = 2.$

Q5 (iii) $\log_6 4 + 2\log_6 3$

$\Rightarrow \log_6 4 + \log_6 9$

$\Rightarrow \log_6 (4\times 9)$

$\Rightarrow \log_6 36 = x$

$6^x = 36$

$x = 2$

Q6 (i) $\log_3 2 + 2\log_3 3 - \log_3 18$

$\Rightarrow \log_3 2 + \log_3 9 - \log_3 18$

$\Rightarrow \log_3 \dfrac{(2\times 9)}{18}$

$\Rightarrow \log_3 1 = 0$

(ii) $\log_8 72 - \log_8 9/8$

$\Rightarrow \log_8 \frac{72}{(9/8)} \Rightarrow \log_8 \left(\frac{72 \times 8}{9}\right)$

$\Rightarrow \log_8 64 = x$

$8^x = 64$

$x = 2.$

Q7 $\boxed{\log_3 5 = a.}$

(i) $\log_3 15 \Rightarrow \log_3 (3 \times 5) \Rightarrow \log_3 3 + \log_3 5$

$= 1 + a.$

(ii) $\log_3 \left(\frac{5}{3}\right) \Rightarrow \log_3 5 - \log_3 3$

$= a - 1$

$\log_3 5 = a$

Q7 (iii)

$\log_3\left(8\tfrac{1}{3}\right) \Rightarrow \log_3\left(\frac{25}{3}\right) \Rightarrow \log_3 \overset{(5\times5)}{25} - \log_3 3$

$\Rightarrow \log_3 5 + \log_3 5 - \log_3 3$

$\Rightarrow 2a - 1$

(iv) $\log_3\left(\dfrac{\overset{5\times5}{25}}{\underset{3\times3\times3}{27}}\right) \Rightarrow \log_3 5 + \log_3 5 - [\log_3 3 + \log_3 3 + \log_3 3]$

$\Rightarrow 2a - 3$

(or)

$\log_3 \frac{25}{27} \Rightarrow \log_3 25 - \log_3 27$

$\log_3 5^2 - \log_3 3^3$

$2\log_3 5 - 3\log_3 3$

$2a - 3(1)$

$2a - 3$

(v) $\log_3 75 \Rightarrow \log_3(25\times3) \Rightarrow 2\log_3 5 + \log_3 3$

$2a + 1$

$2a \qquad 3(1)$

$2a - 3.$

(v) $\log_3 75 \Rightarrow \log_3 (25 \times 3) \Rightarrow 2\log_3 5 + \log_3 3$

$2a + 1$

Q8 (i) $200 = 2^x$

$\log 200 = \log 2^x$

$\log 200 = x \log 2$

$\dfrac{\log 200}{\log 2} = x$

(calculator) $7.64 = x$

(ii) $5^x = 500$

$\log 5^x = \log 500$

$x \log 5 = \log 500$

$x = \dfrac{\log 500}{\log 5}$

$x = 3.86$

Q8 (iii) $3^{x+1} = 25$

$\log 3^{x+1} = \log 25$

$(x+1)\log 3 = \log 25$

$x+1 = \dfrac{\log 25}{\log 3}$

$x+1 = 2.9299$

$x = 1.93$

(iv) $5^{2x+3} = 51$

$(2x+3)\log 5 = \log 51$

$2x+3 = \dfrac{\log 51}{\log 5}$

$2x+3 = 2.443$

$2x = -0.557$

$x = -0.279$

Q9 $y = 2^{x-1} + 3$

(i) $2^{x-1} = y-3$

$\log 2^{x-1} = \log(y-3)$

$x-1 \log 2 = \log(y-3)$

$x-1 = \dfrac{\log(y-3)}{\log_2}$

$x = \dfrac{\log(y-3)}{\log 2} + 1$

$$x-1\log 2 = \log(y-3)$$

$$x-1 = \frac{\log(y-3)}{\log 2}$$

$$x = \frac{\log(y-3)}{\log 2} + 1$$

(ii) $y = 8$

$$x = \frac{\log(8-3)}{\log 2} + 1$$

$$x = \frac{\log 5}{\log 2} + 1$$

$$x = 2.3219 + 1$$

$$x = 3.3219$$

Q10 $\log_{10} x = 1 + a$ $\qquad\qquad$ $\log_{10} y = 1 - a$

Show $xy = 100$

$$10^{1+a} = x \qquad\qquad 10^{1-a} = y$$

$$\Rightarrow \quad xy = \left(10^{1+a}\right)\left(10^{1-a}\right)$$ mult $\Rightarrow$ add powers

$$xy = 10^{1+a+1-a}$$
$$xy = 10^2$$
$$xy = 100.$$

Q11 $P = \log_a\left(\frac{21}{4}\right)$, $q = \log_a\left(\frac{7}{3}\right)$, $r = \log_a\left(\frac{7}{2}\right)$

Show $P + q = 2r$

Q11 $P = \log_a\left(\frac{21}{4}\right)$, $q = \log_a\left(\frac{7}{3}\right)$, $r = \log_a\left(\frac{7}{2}\right)$

Show $P + q = 2r$

$\Rightarrow \log_a\left(\frac{21}{4}\right) + \log_a\left(\frac{7}{3}\right) = 2\log_a\left(\frac{7}{2}\right)$

$\Rightarrow \log_a 21 - \log_a 4 + \log_a 7 - \log_a 3 = 2\log_a\left(\frac{7}{2}\right)$

$\log_a(3\times7) - \log_a(2\times2) + \log_a 7 - \log_a 3$

$\cancel{\log_a 3} + \log_a 7 - (\log_a 2 + \log_a 2) + \log_a 7 - \cancel{\log_a 3}$

$2\log_a 7 - 2(\log_a 2)$

$2(\log_a 7 - \log_a 2)$

$2\log_a\left(\frac{7}{2}\right)$

Hence $P + q = 2r$

Q12 $\log_a x = 4$ $\log_a y = 5$

$\cancel{a^4 = x}$ $\cancel{a^5 = y}$

(i) $\log_a x^2 y$

$\Rightarrow \log_a x^2 + \log_a y$

$= 2\log_a x + \log_a y$

$= 2(4) + 5$

$= 8 + 5$

$= 13$

(ii) $\log_a axy$

$\log_a a + \log_a x + \log_a y$

$1 + 4 + 5$

$= 10$

(iii) $\log_a \frac{\sqrt{x}}{y}$

$\log_a x^{\frac{1}{2}} - \log_a y$

$\frac{1}{2}\log_a x - \log_a y$

$\frac{1}{2}(4) - 5$

$2 - 5 = -3$

Q13 $\log_a x$ ... $\log_a x$

y

$\log_a x^{\frac{1}{2}} - \log_a y$

$\frac{1}{2}\log_a x - \log_a y$

$\frac{1}{2}(4) - 5$

$2 - 5 = -3$

Q B $\quad \log_{25} x = \frac{1}{2}\log_5 x$

$$\log_{25} x = \frac{\log_5 x}{\log_5 25}$$

$$= \frac{\log_5 x}{\log_5 5^2}$$

$\boxed{\log_5 5 = 1}$

$$= \frac{\log_5 x}{2(\log_5 5)} = \frac{\log_5 x}{2} = \frac{1}{2}\log_5 x$$

Q14 (i) $\log_{10} 4 = 0.602$

(ii) $\log_{10} 27 = 1.43$

(iii) $\log_{10} 356 = 2.55$

(iv) $\log_{10} 5600 = 3.75$

(v) $\log_{10} 29000 = 4.46$

(vi) $\log_{10} 350{,}000 = 5.54$

(vii) $\log_{10} 3{,}870{,}000 = 6.59$

Q15 $\log x = 3.123$

Q15 $\log_{10} x = 3.123$

$10^{3.123} = x$

$10^3 = 1000$ (min)
$10^4 = 10{,}000$ (max)

Q16 $\log_3 15 - \log_2 5$

$\frac{\log_{10} 15}{\log_{10} 3} - \frac{\log_{10} 5}{\log_{10} 2}$

$2.46497 - 2.3219 = 0.14307$
$= 0.143$

Q17 (i) $\log_{27} 81 = \frac{\log_3 81}{\log_3 27} = \frac{4}{3}$

(ii) $\log_{32} 8 = \frac{\log_2 8}{\log_2 32} = \frac{3}{5}$

Q18 $\boxed{\log_b a = \frac{1}{\log_a b}}$ NB. Into Notes

Change $\log_b a$ to base a.

$$\frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$$ QED.

Q19 If $x > 0$ & $x \neq 1$ show $\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} = \frac{1}{\log_{30} x}$

Using fact established in Q18.

$$\log_x 2 + \log_x 3 + \log_x 5$$

$$= \log_x (2 \times 3 \times 5)$$

$$= \log_x 30 = \frac{1}{\log_{30} x}$$

QED

Q20 $\quad Log_r P = Log_r 2 + 3 log_r q$ $\quad$ express P in terms of q

$$Log_r P = log_r 2 + log_r q^3$$

$$log_r P = log_r 2q^3$$

$$P = 2q^3$$

Q21 $\quad log_3 a + log_9 a = \frac{3}{4}$ $\quad$ find a

$$log_3 a + \frac{log_3 a}{log_3 9} = \frac{3}{4}$$

$$log_3 a + \frac{log_3 a}{\ldots} = \frac{3}{4}$$

Q21 $\log_3 a + \log_9 a = \frac{3}{4}$ find a

$$\log_3 a + \frac{\log_3 a}{\log_3 9} = \frac{3}{4}$$

$$\log_3 a + \frac{\log_3 a}{\log_3 3^2} = \frac{3}{4}$$

$$\log_3 a + \frac{\log_3 a}{2\log_3 3} = \frac{3}{4}$$

$$\log_3 a + \frac{\log_3 a}{2} = \frac{3}{4} \quad \text{(mult by 2)}$$

$$2\log_3 a + \log_3 a = \frac{3}{2}$$

$$3\log_3 a = \frac{3}{2} \quad \text{(Divide by 3)}$$

$$\log_3 a = \frac{1}{2}$$

$$3^{\frac{1}{2}} = a$$

$$\sqrt{3} = a.$$

Q 22 $3\ln 41.5 - \ln 250$

$3(3.7257) - 5.5215$

$11.177 - 5.5215$

$= 5.6555$

$= 5.66$

[eg 5 + eg 6.]