Ex 3·1

Q1 (i) $y = x^2 - 3x + 2$ at $(1, 0)$

slope $= \dfrac{dy}{dx} = 2x - 3$ at $x = 1$

$$= 2(1) - 3 = -1$$

(ii) $y = x + \dfrac{1}{x}$ at $\left(\tfrac{1}{2}, \tfrac{5}{2}\right)$

$$y = x + x^{-1}$$

$\dfrac{dy}{dx} = 1 - 1x^{-2}$ at $x = \tfrac{1}{2}$

$$= 1 - \left(\tfrac{1}{2}\right)^{-2}$$

$$= 1 - (2)^2 = 1 - 4 = -3$$

Q2 $f(x) = 2x^2 - 4x - 5$ at $(3, 1)$

$f'(x) = 4x - 4$ at $x = 3$

slope $= 12 - 4 = 8$

$$y - 1 = 8(x - 3)$$
$$y - 1 = 8x - 24$$
$$8x - y - 23 = 0$$

Q3

$$f(x) = x^2 - 6x \qquad \text{at} \quad x = 2$$
$$f'(x) = 2x - 6$$
$$\text{slope} = 2(2) - 6 = -2$$

Find y co-ord: $\quad y = x^2 - 6x \quad$ but $x = 2$
$$\Rightarrow \quad y = (2)^2 - 6(2) = -8.$$

$\Rightarrow$ Eqn of Tangent with $m = -2$ and pt $(2, -8)$

$$y + 8 = -2(x - 2)$$
$$y + 8 = -2x + 4$$
$$2x + y + 4 = 0$$

Q5

$$y = x^2 + kx \qquad \text{slope is } 3 \quad \text{at } x = -1$$
$$\Rightarrow \quad \frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = 2x + k$$
$$2(-1) + k = 3$$
$$k = 5$$

Q8

$$y = \frac{5x^2}{1 + x^2} \qquad \text{at} \qquad (2, 4)$$

$$\frac{dy}{dx} = \frac{(1 + x^2)(10x) - (5x^2)(2x)}{(1 + x^2)^2}$$

at $x = 2 \quad \dfrac{(1 + 2^2)(10(2)) - (5(2)^2)(2(2))}{(1 + (2)^2)^2}$

$$\frac{100 - 80}{25} = \frac{20}{25} = \frac{4}{5} = \text{Slope}$$

Eqn at $(2, 4)$ and $m = \frac{4}{5}$
$$y - 4 = \frac{4}{5}(x - 2)$$
$$5y - 20 = 4x - 8$$
$$4x - 5y + 12 = 0$$

Q9   slope $= 0$ $\Rightarrow$ $dy/dx = 0$

$$y = x^3 - 12x + 4$$
$$\frac{dy}{dx} = 3x^2 - 12 = 0$$

$$x^2 - 4 = 0$$
$$(x+2)(x-2) = 0$$
$$x = -2 \qquad x = 2$$

find y co-ords.
$$y = x^3 - 12x + 4$$
at $x = -2$ $y = (-2)^3 - 12(-2) + 4$ at $x - 2$
$$y = 20 \qquad\qquad y = (2)^3 - 12(2) + 4$$
$$y = -12$$
$$(-2, \ 20) \qquad\qquad (2, -12)$$

Q10 $y = ax^2 + bx + 5$ slope $= 4$ at $(5,0)$

$$\frac{dy}{dx} = 2ax + b = 4 \qquad x = 5$$
$$\boxed{10a + b = 4}$$

$(5,0)$ $y = ax^2 + bx + 5$
$$0 = a(5)^2 + b(5) + 5$$
$$-5 = 25a + 5b$$
$$\boxed{5a + b = -1}$$

$$10a + b = 4$$
$$\ominus \ 5a + b = -1 \ \oplus$$
$$\overline{5a = 5}$$
$$\underline{a = 1}$$
$$10(1) + b = 4$$
$$\underline{b = -6}$$

Q12 $\qquad y = \ln x + x - 2$

$\qquad \dfrac{dy}{dx} = \dfrac{1}{x} + 1 \qquad$ at $\;x = 1$

$\qquad \dfrac{1}{1} + 1 = 2 \rightarrow \underline{\text{slope.}}$

$\qquad x = 1 \qquad y = \ln(1) + 1 - 2$

$\qquad\qquad\qquad\qquad = 0 + 1 - 2 = -1 \qquad \underline{pt\,(1, -1)}$

$\qquad$ eqn of Tangent: $\quad y + 1 = 2(x - 1)$

$\qquad\qquad\qquad\qquad\qquad\quad y + 1 = 2x - 2$

$\qquad\qquad\qquad\qquad\qquad\qquad 2x - y - 3 = 0.$

Q13 $\qquad y = e^{3x} \qquad (0, 1)$

$\qquad \dfrac{dy}{dx} = e^{3x} \cdot 3 \;= 3 e^{3x}$

$\qquad\qquad\qquad$ at $x = 0 \;\Rightarrow\; 3 e^{3(0)} = 3 \;\underline{\text{slope}}$

$\qquad$ Tangent: $\quad y - 1 = 3(x - 0)$

$\qquad\qquad\qquad\qquad\; y - 1 = 3x$

$\qquad\qquad\qquad\qquad\quad 3x - y + 1 = 0$

Q14
$$y = x^3 - 3x^2 - 5x + 10$$

Tangents parallel to $y = 4x - 7 \implies$ Slope = 4

$$y = x^3 - 3x^2 - 5x + 10$$
$$\frac{dy}{dx} = 3x^2 - 6x - 5$$

$$3x^2 - 6x - 5 = 4$$
$$3x^2 - 6x - 9 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = 3 \qquad x = -1$$

find y : $y = (3)^3 - 3(3)^2 - 5(3) + 10$
$$y = -5$$
$$\text{Point } (3, -5)$$
$$y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$$
$$y = 11$$
$$\text{Point } (-1, 11)$$

Q15 (1) $y = \frac{-x^2}{125}$

at sea level $y = -500$
$$\overset{\oplus}{-} 500 = \frac{\overset{\oplus}{-} x^2}{125}$$

$$62500 = x^2$$
$$\underline{250 = x}$$

(ii) $\tan \theta = $ Slope $\qquad y = \frac{-x^2}{125}$

$$\frac{dy}{dx} = \frac{-2x}{125}$$

at $x = 250$ $\implies \frac{-2(250)}{125} = -4$ slope

$\tan \theta = -4 \longrightarrow \theta = \tan^{-1}(-4) \implies \theta = 76^\circ$

Q16 increasing $\Rightarrow \frac{dy}{dx} > 0$ decreasing $\Rightarrow \frac{dy}{dx} < 0$

(a) $y = x^2 - x - 6$

$\frac{dy}{dx} = 2x - 1$

decreasing $\Rightarrow$ $2x - 1 < 0$

$2x < 1$

$x < \frac{1}{2}$

(b) $y = x^3 + 6x^2 - 2$
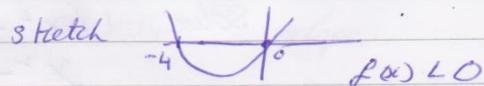
$\frac{dy}{dx} = 3x^2 + 12x$

decreasing $\Rightarrow$ $3x^2 + 12x < 0$

$x^2 + 4x < 0$

$x(x+4) = 0$

$x = 0$ $x = -4$

sketch

$f(x) < 0$

$\Rightarrow -4 < x < 0$

Q17 $f(x) = 4x^2 + 4x + 7$

(i) $f'(x) = 8x + 4$.

(ii) ⓐ increasing $8x + 4 > 0$

$8x > 4$

$x > \frac{1}{2}$
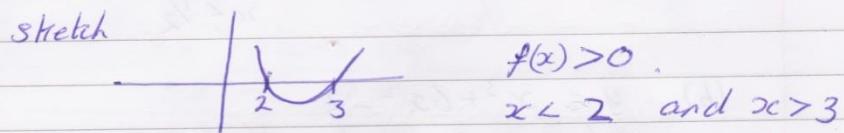
ⓑ decreasing $8x + 4 < 0$

$8x < 4$

$x < \frac{1}{2}$

Q18 (iii)
$$f(x) = 2x^3 - 15x^2 + 36x$$
$$f'(x) = 6x^2 - 30x + 36.$$
solve $6x^2 - 30x + 36 = 0$
$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0$$
$$x = 3 \qquad x = 2$$
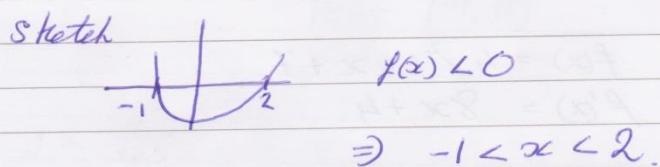
Increasing $\Rightarrow \frac{dy}{dx} > 0$

sketch

$f(x) > 0$.
$x < 2$ and $x > 3$

Q19 (iii)
$$f(x) = 2x^3 - 3x^2 - 12x$$
$$f'(x) = 6x^2 - 6x - 12$$

decreasing $\Rightarrow \frac{dy}{dx} < 0$

Solve $6x^2 - 6x - 12 = 0$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \qquad x = -1$$

sketch

$f(x) < 0$

$\Rightarrow -1 < x < 2$.

Q20 $f(x) = x^3 - 6x^2 + 18x + 4$ $\left[\text{Inc} \Rightarrow \frac{dy}{dx} > 0 \right]$

$$f'(x) = 3x^2 - 12x + 18$$
show $f'(x)$ is always > 0
$$3x^2 - 12x + 18 > 0$$
$$x^2 - 4x + 6 > 0$$
complete the sq $\quad x^2 - 4x + 4 + 6 - 4 > 0$
$$(x - 2)^2 + 2 > 0 \quad \text{True.}$$

Q21 $y = \dfrac{2x+1}{3x+6}$ Increasing $\Rightarrow \dfrac{dy}{dx} > 0$

$$\dfrac{dy}{dx} = \dfrac{(3x+6)(2) - (2x+1)(3)}{(3x+6)^2}$$

$$= \dfrac{6x+12 - 6x - 3}{(3x+6)^2}$$

$$= \dfrac{9}{(3x+6)^2} > 0 \quad \text{True}$$

Q22

(i) $y = x^2 - 14x + 53$ $P(8,5)$

  $dy/dx = 2x - 14$

   at $x = 8$ $2(8) - 14 = 2$ slope

Tangent : $y - 5 = 2(x - 8)$

     $y - 5 = 2x - 16$

      $2x - y - 11 = 0$

(ii) $y = -x^2 + 10x - 27$

  $\dfrac{dy}{dx} = -2x + 10$

   $-2x + 10 = 2$

    $-2x = -8$

     $x = 4$

at $x = 4$ $y = -(4)^2 + 10(4) - 27 = -3$

    $Q(4, -3)$

Q23 $y = 2 + 0.12x - 0.01x^3$

$$0 \leq x \leq 3$$

$$\frac{dy}{dx} = 0.12 - 0.03x^2$$

at $x = 0$ $0.12 - 0.03(0)^2 = 0.12$.

at $x = 3$ $0.12 - 0.03(3)^2 = -0.15$

(ii)   slope $= 0$ $\Rightarrow$ $0.12 - 0.03x^2 = 0$

$$12 - 3x^2 = 0$$

$$4 - x^2 = 0$$

$$(2 + x)(2 - x) = 0$$

$$x = -2 \qquad x = 2$$

But $0 \leq x \leq 3$ $\Rightarrow$ $x = 2$

$$y = 2 + 0.12(2) - 0.01(2)^3$$

$$y = 2.16.$$

$$\Rightarrow \text{ height } = 2.16 \text{ km}$$

©24 $y = \sqrt{x+2}$ meets x at A,   meets y at B

(a) find A.   Cuts $x \Rightarrow y=0$ $0 = \sqrt{x+2}$
                                  $0 = x+2$
                              $-2 = x$ $A(-2, 0)$

find B.   cut y at $x=0$ $y = \sqrt{0+2}$
                                 $y = \sqrt{2}$ $B(0, \sqrt{2})$

(b) $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$
         $dy/dx = \frac{1}{2}(x+2)^{-\frac{1}{2}}(1)$
                $= \frac{1}{2\sqrt{x+2}}$

(c) (i) gradient where $x = -1$ $\Rightarrow$ $\frac{1}{2\sqrt{-1+2}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

(ii) Tangent :   Slope $= \frac{1}{2}$ pt $(-1, ?)$
                         $y = \sqrt{-1+2} = 1$
                     $\Rightarrow$ pt $(-1, 1)$
      eqn: $y - 1 = \frac{1}{2}(x+1)$
              $2y - 2 = x+1$
              $x - 2y + 3 = 0$

(iii) meet x axis at C $\Rightarrow$ $y = 0$
                 $x - 2(0) + 3 = 0$ $\overset{x_1 \; y_1}{}$
                       $x = -3$ $C(-3, 0)$
   meets y axis at D $\Rightarrow$ $x = 0$
                 $0 - 2y + 3 = 0$ $\overset{x_2 \; y_2}{}$
                      $y = \frac{3}{2}$ $D(0, \frac{3}{2})$

$|CD| = \sqrt{(0+3)^2 + (\frac{3}{2}-0)^2} = \sqrt{9 + \frac{9}{4}} = \sqrt{\frac{45}{4}}$
                                              $= \frac{3\sqrt{5}}{2}$

(d) $\dfrac{dy}{dx} < 1$

$$\frac{1}{2\sqrt{x+2}} < 1$$

$$1 < 2\sqrt{x+2}$$

$$\frac{1}{2} < \sqrt{x+2}$$

$$\frac{1}{4} < x+2$$

$$\frac{1}{4} - 2 < x$$

$$-\frac{7}{4} < x$$