Ex 3.10

Q1 (i) $(\cos\pi - i\sin\pi)^5$

$= (\cos -\pi + i\sin -\pi)^5 = \cos(-5\pi) + i\sin(-5\pi)$

$= -1 + 0i$

(ii) $(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5})^{10}$

$= (\cos -\frac{\pi}{5} + i\sin -\frac{\pi}{5})^{10} = \cos(-2\pi) + i\sin(-2\pi)$

$= 1 + 0i$

(iii) $\dfrac{1}{(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^3} = (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})^{-3}$

$= (\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3})^{-3} = \cos\pi + i\sin\pi$

$= -1 + 0i$

(iv) $(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^4 = (\cos -\frac{\pi}{2} + i\sin -\frac{\pi}{2})^4$

$(\cos -2\pi + i\sin -2\pi) = 1 + 0i$

Q2 (i) $\sin 2\theta$: $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$

$\cos 2\theta + i\sin 2\theta = \cos^2\theta + 2\cos\theta\sin\theta i - \sin^2\theta$

$\cos 2\theta + i\sin 2\theta = \cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta i$

equate: $\sin 2\theta = 2\cos\theta\sin\theta$

(~~ii~~)

Q2 (ii) $\cos 3\theta$:

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3\cos^2\theta\sin\theta\, i \overset{-}{+} 3\cos\theta\sin^2\theta\,\cancel{i^2} \overset{-}{+} \sin^3\theta\,\cancel{i^2}i$$

$$\cos 3\theta + i\sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + (3\cos^2\theta\sin\theta - \sin^3\theta)i$$

equate:
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\cos 3\theta = \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$$
$$\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Q3

(i) $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$
$$= \cos^4\theta + 4\cos^3\theta\sin\theta\, i \overset{-}{+} 6\cos^2\theta\sin^2\theta\,\cancel{i^2}$$
$$\overset{-}{+} 4\cos\theta\sin^3\theta\,\cancel{i^2}i + \sin^4\theta\,\cancel{i^4}$$

equate real:
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$ Q.E.D.

(ii) equate img:
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$
Q.E.D.

Q4 $z^3 = 8$

$z = 8^{1/3}$

$8 + 0i$ $\quad r = \sqrt{8^2 + 0^2} = 8.$

$\theta = 0.$

$$8 + 0i = 8(\cos 0 + i \sin \theta)$$

$$= 8(\cos(0 + 2n\pi) + i \sin(0 + 2n\pi))$$

$$\therefore 8^{1/3} = 8^{1/3}\left(\cos \tfrac{1}{3}(0 + 2n\pi) + i \sin \tfrac{1}{3}(0 + 2n\pi)\right)$$

Let $n = 0$ $\quad = 2(\cos \theta + i \sin 0)$

$= 2(1 + 0i)$

$= 2 + 0i$

Let $n = 1$ $\quad = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$

$= 2\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} i\right)$

$= -1 + \sqrt{3} i$

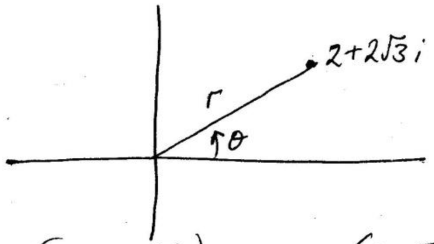
Let $n = 2$ $\quad = 2\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

$= 2\left(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2} i\right)$

$= -1 - \sqrt{3} i$

$\therefore$ Roots are $z = 2 + 0i$ and $-1 + \sqrt{3} i$ and $-1 - \sqrt{3} i$

Q6 $\quad 2+2\sqrt{3}\,i$



$r=\sqrt{2^2+(2\sqrt{3})^2}=4$

$\theta=\tan^{-1}(\sqrt{3})=\pi/3$

$$\Rightarrow (2+2\sqrt{3}i) = 4(\cos \pi/3 + i\sin \pi/3)$$

$$z^2 = 2+2\sqrt{3}i \quad \Rightarrow \quad z = (2+2\sqrt{3}i)^{1/2}$$

$$= 4^{\frac{1}{2}}\left(\cos \tfrac{1}{2}\left(\tfrac{\pi}{3}+2\pi n\right) + i\sin \tfrac{1}{2}\left(\tfrac{\pi}{3}+2\pi n\right)\right)$$

let $n=0$

$$= 2(\cos \pi/6 + i\sin \pi/6)$$

$$= 2\left(\sqrt{3}/2 + \tfrac{1}{2}i\right)$$

$$= \sqrt{3}+i$$

let $n=1 \Rightarrow 2\left(\cos \tfrac{7\pi}{6} + i\sin 7\pi/6\right)$

$$= 2\left(-\sqrt{3}/2 - 1/2\, i\right)$$

$$= -\sqrt{3}-i$$

$\therefore$ Roots are $z = \sqrt{3}+i$ and $-\sqrt{3}-i$

Q7 $\quad z = 1 \qquad |z| = 1 \qquad \theta = 0$ radians

(a) $z = 1\left(\cos(0+2n\pi) + i\sin(0+2n\pi)\right)$

$z = 1^{1/3}$

$\Rightarrow 1^{1/3}\left(\cos\frac{1}{3}(0+2n\pi) + i\sin\frac{1}{3}(0+2n\pi)\right)$

Let $n=0 \Rightarrow 1(\cos 0 + i\sin 0)$
$= 1 + 0i$

Let $n=1 \Rightarrow 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= -1/2 + \sqrt{3}/2\, i$

Let $n=2 \Rightarrow 1\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$
$= -1/2 - \sqrt{3}/2\, i$

$\therefore$ Roots are: $1+0i$, $-\frac{1}{2} + \sqrt{3}/2\, i$, $-\frac{1}{2} - \sqrt{3}/2\, i$

(b) Sum equals zero

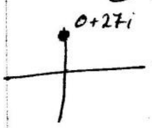
$(1+0i) + \left(-\frac{1}{2} + \sqrt{3}/2\, i\right) + \left(-1/2 - \sqrt{3}/2\, i\right)$

$= \left(1 - 1/2 - \frac{1}{2}\right) + \left(0 + \sqrt{3}/2 - \sqrt{3}/2\right) i$

$= 0 + 0i \quad = 0.$

Q8 cube Roots of $27i$

$0+27i \qquad r = \sqrt{0^2+27^2} = 27$



$\theta = \pi/2$

$$0+27i = 27\left(\cos\left(\tfrac{\pi}{2}+2n\pi\right) + i\sin\left(\tfrac{\pi}{2}+2n\pi\right)\right)$$

Cubic Roots

$$= 27^{\frac{1}{3}}\left(\cos\tfrac{1}{3}\left(\tfrac{\pi}{2}+2n\pi\right) + i\sin\tfrac{1}{3}\left(\tfrac{\pi}{2}+2n\pi\right)\right)$$

Let $n=0 \Rightarrow 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 3\left(\sqrt{3}/2 + \frac{1}{2}i\right)$

$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

Let $n=1 \Rightarrow 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -\frac{3\sqrt{3}}{2} + \frac{3}{2}i$

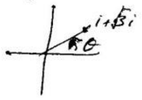
Let $n=2 \Rightarrow 3\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right)$

$3(0 - 1i)$

$= 0 - 3i$

$\therefore$ Roots are: $\frac{3\sqrt{3}}{2} + \frac{3}{2}i, \quad -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, \quad 0-3i$

Q9 (i) $z^2 = 1 + \sqrt{3}i$
$z = (1+\sqrt{3}i)^{1/2}$



$r = \sqrt{1^2 + \sqrt{3}^2} = 2.$
$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$$(1+\sqrt{3}i)^{1/2} = 2^{1/2}\left(\cos\left(\tfrac{\pi}{3}+2n\pi\right) + i\sin\left(\tfrac{\pi}{3}+2n\pi\right)\right)^{1/2}$$

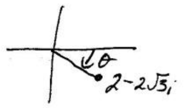
$$= \sqrt{2}\left(\cos\tfrac{1}{2}\left(\tfrac{\pi}{3}+2n\pi\right) + i\sin\tfrac{1}{2}\left(\tfrac{\pi}{3}+2n\pi\right)\right)$$

Let $n=0$ $\Rightarrow \sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
$= \sqrt{2}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$
$= \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$

Let $n=1$ $\Rightarrow \sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$
$= \sqrt{2}\left(-\sqrt{3}/2 - \frac{1}{2}i\right)$
$= -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

$\boxed{\frac{1}{2}\left(\frac{\pi}{3}+2\pi\right) \quad \frac{1}{2}\left(\frac{7\pi}{3}\right) = \frac{7\pi}{6}}$

(ii) $z^2 = 2 - 2\sqrt{3}i$
$z = (2-2\sqrt{3}i)^{1/2}$



$r = \sqrt{2^2 + (-2\sqrt{3})^2} = 4$
$\theta = \tan^{-1}(-\sqrt{3}) = -\pi/3$

$$(2-2\sqrt{3}i)^{1/2} = 4^{1/2}\left(\cos\left(-\tfrac{\pi}{3}+2n\pi\right) + i\sin\left(-\pi/3+2n\pi\right)\right)^{1/2}$$

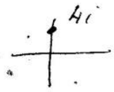
$$= 2\left(\cos\tfrac{1}{2}\left(-\tfrac{\pi}{3}+2n\pi\right) + i\sin\tfrac{1}{2}\left(-\pi/3+2n\pi\right)\right)$$

Let $n=0$ $\Rightarrow 2\left(\cos -\pi/6 + i\sin -\pi/6\right)$
$2\left(\sqrt{3}/2 - 1/2\,i\right) = \sqrt{3} - i$

Let $n=1$ $\Rightarrow 2\left(\cos 5\pi/6 + i\sin 5\pi/6\right)$
$= 2\left(-\sqrt{3}/2 + \frac{1}{2}i\right)$
$= -\sqrt{3} + i$

$\boxed{-\pi/3 + 2\pi = \frac{5\pi}{3} \quad \frac{1}{2}\left(\frac{5\pi}{3}\right) = \frac{5\pi}{6}}$

$\Rightarrow$ roots are: $\sqrt{3} - i$ and $-\sqrt{3} + i$

(iii) $z^2 = 4i$  $r = 4 \quad \theta = \pi/2$

$z = (4i)^{\frac{1}{2}}$

$(4i)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(\cos(\pi/2 + 2n\pi) + i\sin\left(\frac{\pi}{2} + 2n\pi\right)\right)^{\frac{1}{2}}$

Let $n=0 \Rightarrow 2\left(\cos\frac{1}{2}\left(\frac{\pi}{2}\right) + i\sin\frac{1}{2}\left(\frac{\pi}{2}\right)\right)$

$= 2\left(\sqrt{2}/2 + \sqrt{2}/2\, i\right)$

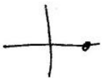
$= \sqrt{2} + \sqrt{2}\, i$

Let $n=1 \Rightarrow 2\left(\cos\frac{1}{2}\left(\frac{5\pi}{2}\right) + i\sin\frac{1}{2}\left(5\pi/2\right)\right)$

$= 2\left(-\sqrt{2}/2 - \sqrt{2}/2\, i\right)$

$= -\sqrt{2} - \sqrt{2}\, i$

Roots are: $\sqrt{2} + \sqrt{2}i$ and $\sqrt{2} - \sqrt{2}i$

Q10 $z^5 = 1$  $r = 1 \quad \theta = 0$

$z =$

$(1+0i)^{\frac{1}{5}} = 1^{\frac{1}{5}}\left(\cos(0+2n\pi) + i\sin(0+2n\pi)\right)^{\frac{1}{5}}$

$= 1\left(\cos\frac{1}{5}(0+2n\pi) + i\sin\frac{1}{5}(0+2n\pi)\right)$

Let $n=0 \Rightarrow (\cos 0 + i\sin 0) = 1 + 0i = 1$

Let $n=1 \Rightarrow \left(\cos\frac{2\pi}{5} + i\sin 2\pi/5\right) = 0.309 + 0.951i$

Let $n=2 \Rightarrow \left(\cos\frac{4\pi}{5} + i\sin 4\pi/5\right) = -0.809 + 0.588i$

Let $n=3 \Rightarrow \left(\cos 6\pi/5 + i\sin 6\pi/5\right) = -0.809 - 0.588i$

Let $n=4 \Rightarrow \cos\left(8\pi/5 + i\sin 8\pi/5\right) = 0.309 - 0.951i$

Let $w = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) \Rightarrow w^2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$w^3 = \left(\cos\frac{6\pi}{5} + i\sin 6\pi/5\right) \cdot w^2 + w^3 = -1.618 + 0i$ = Real!