Ex 3.2

Q5 (i) $y = 2x^3 - 3x^2 - 12x + 5$

$\frac{dy}{dx} = 6x^2 - 6x - 12 = 0 \quad (\div 6)$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2 \quad x = -1$

find y:

at $x = 2$: $y = 2(2)^3 - 3(2)^2 - 12(2) + 5 = -15$

$(2, -15)$

at $x = -1$: $y = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5 = 12$

$(-1, 12)$

$\frac{d^2y}{dx^2} = 12x - 6$

at $x = 2$ $\quad 12(2) - 6 = 18$ Pos $\Rightarrow (2, -15)$ Min.

at $x = -1$ $\quad 12(-1) - 6 = -18$ Neg $\Rightarrow (-1, 12)$ Max.

(ii) $y = \frac{x^2}{x+2}$

$\frac{dy}{dx} = \frac{(x+2)(2x) - (x^2)(1)}{(x+2)^2} = \frac{2x^2 + 4x - x^2}{(x+2)^2} = \frac{x^2+4x}{(x+2)^2}$

$\frac{dy}{dx} = 0 \qquad \frac{x^2+4x}{(x+2)^2} = 0$

$x^2 + 4x = 0$

$x(x+4) = 0$

$x = 0 \qquad x = -4$

find y

at $x = 0$ $\quad y = \frac{0^2}{0+2} = 0 \quad (0,0)$

at $x = -4$ $\quad y = \frac{(-4)^2}{(-4)+2} = \frac{16}{-2} = -8 \quad (-4, -8)$

find $\frac{d^2y}{dx^2}$ $\qquad dy/dx = \frac{x^2+4x}{(x+2)^2}$

$$d^2y/dx^2 = \frac{(x+2)^2(2x+4) - (x^2+4x)2(x+2)(1)}{(x+2)^4}$$

at $x=0$ $$\frac{d^2y}{dx^2} = \frac{(0+2)^2(2(0)+4) - ((0)^2+4(0))2(0+2)(1)}{(0+2)^4}$$

$$= \frac{16-0}{16} = 1 \text{ pos} \Rightarrow (0,0) \text{ Min}$$

at $x = -4$ $$\frac{d^2y}{dx^2} = \frac{(\overset{4}{-4+2})^2(\overset{-4}{2(-4)+4}) - ((\overset{16}{-4})^2 + \overset{-16}{4(-4)})2(\overset{-2}{-4+2})(1)}{(-4+2)^4}$$

$$= \frac{-16-0}{16} = \frac{-16}{16} = -1 \Rightarrow (-4, -8) \text{ Max}$$

Q8 $$y = x - \sqrt{x}$$
$$y = x - x^{\frac{1}{2}}$$

$$dy/dx = 1 - \frac{1}{2}x^{-\frac{1}{2}}$$
$$= 1 - \frac{1}{2\sqrt{x}} = 0$$
$$2\sqrt{x} - 1 = 0$$
$$\sqrt{x} = \frac{1}{2}$$
$$x = \frac{1}{4}$$

find y: $y = \frac{1}{4} - \sqrt{\frac{1}{4}}$
$$= \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$
$$\left(\frac{1}{4}, -\frac{1}{4}\right)$$

$dy/dx = 1 - \frac{1}{2}x^{-\frac{1}{2}}$

$$\frac{d^2y}{dx^2} = +\frac{1}{4}x^{-3/2}$$
$$= \frac{1}{4\sqrt{x^3}}$$

at $x = \frac{1}{4}$

$$\frac{1}{4\sqrt{\frac{1}{4}^3}} = \frac{1}{4\sqrt{\frac{1}{64}}}$$
$$= \frac{1}{4\left(\frac{1}{8}\right)} = \frac{1}{\frac{1}{2}} = 2.$$

pos $\Rightarrow \left(\frac{1}{4}, -\frac{1}{4}\right)$ is a Min pt.

Q9 (i) $y = x^3 + 3x^2 + 1$

$dy/dx = 3x^2 + 6x$

$\frac{d^2y}{dx^2} = 6x + 6 = 0$

$6x = -6$

$x = -1$

find y : $y = (-1)^3 + 3(-1)^2 + 1 = 3$

$\Rightarrow$ Pt of Inflection is $(-1, 3)$

(ii)

$y = x^3 - 6x^2 + 9x + 2$

$dy/dx = 3x^2 - 12x + 9$

$d^2y/dx^2 = 6x - 12 = 0$

$6x = 12$

$x = 2.$

find y: $y = (2)^3 - 6(2)^2 + 9(2) + 2 = 4$

$\Rightarrow$ Pt of Inflection is $(2, 4)$

Q10 $y = \cos x$

$dy/dx = -\sin x$

$\frac{d^2y}{dx^2} = -\cos x$

at $x = \pi/2 \Rightarrow -\cos \pi/2 = -0 = 0.$

$\therefore$ Pt of Inflection is at $x = \pi/2$

Q11 $\quad y = ax^3 + bx^2 + c$.

$dy/dx = 3ax^2 + 2bx = 0$ at $(0,4)$
$\Rightarrow x = 0$

$3a(0)^2 + 2b(0) = 0$
$0 = 0$

at $(-1,5)$
$\Rightarrow x = -1$

$3a(-1)^2 + 2b(-1) = 0$
$\boxed{3a - 2b = 0}$

$(0,4)$ is on $y = ax^3 + bx^2 + c$
$4 = a(0)^3 + b(0)^2 + c$
$4 = c$

$(-1,5)$ is on $y = ax^3 + bx^2 + 4$
$5 = a(-1)^3 + b(-1)^2 + 4$
$\boxed{1 = -a + b}$ $(\times 3)$

$3a - 2b = 0$
$-3a + 3b = 3$
$b = 3$

$1 = -a + 3$
$a = 2$

$a = 2, \quad b = 3, \quad c = 4$.

Q13 $\quad y = 2x^2 - \ln x$

Slope $= dy/dx = 4x - \frac{1}{x}$

at $x = 1 \quad 4(1) - \frac{1}{1} = 4 - 1 = 3$ QED

(ii) Turning pt $\Rightarrow dy/dx = 0$

$$4x - \frac{1}{x} = 0$$

$$4x^2 - 1 = 0$$

$$(2x+1)(2x-1) = 0$$

$$x = -1/2 \quad x = 1/2$$

find y:

at $x = 1/2 \quad y = 2\left(\frac{1}{2}\right)^2 - \ln\frac{1}{2}$

$$= \frac{1}{2} - \ln 2^{-1}$$

$$= 1/2 + \ln 2$$

~~at $x = -1/2 \quad y = 2\left(\frac{-1}{2}\right)^2 - \ln\left(\frac{-1}{2}\right)$~~ Since $x > 0$.

~~$y = \frac{1}{4}$~~

$\frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = 4x - \frac{1}{x} = 4x - x^{-1}$$

$$\frac{d^2y}{dx^2} = 4 + x^{-2} = 4 + \frac{1}{x^2}$$

at $x = 1/2 \quad 4 + \frac{1}{(1/2)^2}$

$$= 4 + \frac{1}{1/4}$$

$$= 4 + 4 = 8 > 0 \Rightarrow \text{Pos} \Rightarrow \text{Min Pt.}$$

Q14 $y = e^{x} - x$

(i) $\frac{dy}{dx} = e^{x} - 1 = 0$

$e^{x} = 1$

$\Rightarrow x = 0.$

find y: $y = e^{0} - 0 = 1 - 0 = 1$ $(0, 1)$.

$\frac{dy}{dx} = e^{x} - 1$

$\frac{d^2y}{dx^2} = e^{x}$

at $x = 0$ $e^{0} = 1 > 0$ Pos $\Rightarrow$ Min pt.

Q15 $f(x) = (1+x)\log_e(1+x)$ $x > -1$

$f'(x) = \cancel{(1+x)}\frac{1}{\cancel{1+x}} + \log_e(1+x)(1)$

$1 + \log_e(1+x) = 0$

$\log_e(1+x) = -1$

$1 + x = e^{-1}$

$1 + x = \frac{1}{e}$

$x = \frac{1}{e} - 1 = \frac{1-e}{e}$

find y:

$y = \left(1 + \frac{1-e}{e}\right)\log_e\left(1 + \frac{1-e}{e}\right)$

$= \frac{1}{e}\log_e\frac{1}{e}$

$= \frac{1}{e}\log_e e^{-1}$

$= -\frac{1}{e}\log_e e$

$= \frac{-1}{e}.$

$\left(\frac{1-e}{e}, \frac{-1}{e}\right)$

(ii) $\frac{dy}{dx} = 1 + \log_e(1+x)$

$\frac{d^2y}{dx^2} = \frac{1}{1+x}$

at $x = \left(\frac{1}{e} - 1\right)$ $\frac{d^2y}{dx^2} = \frac{1}{1+\left(\frac{1}{e}-1\right)} = \frac{1}{1/e} = e > 0$

pos $\Rightarrow$ Min Pt.

Q17 $f(x) = ax^3 + bx^2 + cx + d$ Max at $(0,4)$

Inflec at $(1,0)$

Max $\Rightarrow \frac{dy}{dx} = 0$ at $x = 0$

$\frac{dy}{dx} = 3ax^2 + 2bx + c = 0$

at $x=0$ $3a(0)^2 + 2b(0) + c = 0$

$c = 0$

Inflection $\Rightarrow \frac{d^2y}{dx^2} = 0$ at $x = 1$

$\frac{d^2y}{dx^2} = 6ax + 2b = 0$

at $x=1$ $6a + 2b = 0$

$(0,4)$ $f(x)$: $4 = a(0)^3 + b(0)^2 + c(0) + d$

$d = 4$

$(1,0)$ $f(x)$: $0 = a(1)^3 + b(1)^2 + c(1) + 4$

$-4 = a + b + c$ but $c = 0$

$-4 = a + b$

$a + b = -4$

$6a + 2b = 0$ $(\div 2)$

$\ominus$ $a + b = -4$

$3a + b = 0$

$2a = 4$

$a = 2$

$2 + b = -4$

$b = -6$

Q18 $f(x) = \frac{x}{x+2}$ $x \neq -2$

Turning Pt's $\Rightarrow f'(x) = 0$

$$f'(x) = \frac{(x+2)(1) - x(1)}{(x+2)^2} = \frac{x+2-x}{(x+2)^2} = \frac{2}{(x+2)^2}$$

cannot $= 0$ as Numerator is not zero

$\Rightarrow$ No turning Pts

Pt of Inflection $\Rightarrow f''(x) = 0$

$$f''(x) = \frac{(x+2)^2(0) - 2(x+2)^2(1)}{(x+2)^4}$$

$$\frac{0 - 2(x+2)^2}{(x+2)^{4\,2}} = \frac{-2}{(x+2)^2} \neq 0$$

$\Rightarrow$ No pt of Inflection

Q19 $g(x) = x^2 + \frac{a}{x^2} = x^2 + ax^{-2}$

Turning pt at $x = 2$.

$$g'(x) = 2x - 2ax^{-3} = 2x - \frac{2a}{x^3}$$

at $x = 2$

$$2(2) - \frac{2}{(2)^3} = 0$$

$$4 - \frac{2a}{8} = 0$$

$$32 - 2a = 0$$

$$2a = 32$$

$$\underline{a = 16}$$

Q19 (ii) $g(x)$ has no local Max

$$g(x) = x^2 + \frac{16}{x^2} = x^2 + 16x^{-2}$$

Max pt $\Rightarrow \frac{dy}{dx} = 0$

$$g'(x) = 2x - 32x^{-3}$$

$$2x - \frac{32}{x^3} = 0$$

$$2x^4 - 32 = 0$$

$$x^4 - 16 = 0$$

$$(x^2 - 4)(x^2 + 4) = 0$$

$x^2 - 4 = 0$ $\quad$ $x^2 + 4 = 0$

$(x+2)(x-2) = 0$ $\quad$ $\cancel{x^2 = -4}$

$x = -2$ $\quad$ $x = 2$

Test for Max/Min $\quad \frac{d^2y}{dx^2} = 0$

$$g'(x) = 2x - 32x^{-3}$$

$$g''(x) = 2 + 96x^{-4} = 2 + \frac{96}{x^4}$$

at $x = -2$ $\quad g''(x) = 2 + \frac{96}{(-2)^4} = 8 > 0$ pos

$\Rightarrow$ a Min Pt

at $x = 2$ $\quad g''(x) = 2 + \frac{96}{(2)^4} = 8 > 0$ pos

$\Rightarrow$ a Min Pt

$\therefore$ $g(x)$ has no Max pt.

Q20 $C = \frac{1400}{v} + \frac{2v}{7}$

(i) C is a min $\Rightarrow \frac{dc}{dv} = 0$

$C = 1400v^{-1} + \frac{2}{7}v$

$dc/dv = -1400v^{-2} + \frac{2}{7} = 0$

$= \frac{-1400}{v^2} + \frac{2}{7} = 0 \quad (\times 7v^2)$

$-9800 + 2v^2 = 0$

$v^2 = 4900$

$v = \pm 70$

v = speed $\Rightarrow$ $v = 70$ km/hr

(ii)

$\frac{dc}{dv} = -1400v^{-2} + \frac{2}{7}$

$\frac{d^2c}{dv^2} = 2(1400)v^{-3} = \frac{2800}{v^3}$

at $v = 70$ $\quad \frac{2800}{(70)^3} = \frac{2800}{343000} > 0$ Pos $\Rightarrow$ Min Pt

(iii)

at $v = 70$ $\quad C = \frac{1400}{70} + \frac{2(70)}{7}$

$= 20 + 20$

$= €40$