Revision Exercise (Core)

Q1 $\sqrt{80} - \sqrt{20} = 4\sqrt{5} - 2\sqrt{5} = 2\sqrt{5}$

Q2 $(x-1) + yi = y + 4i$

RE: $x - 1 = y$, $x - 1 = 4$, $\underline{\underline{x = 5}}$

im: $\underline{\underline{y = 4}}$

Q3 $z^2 + 4z + 3 = 0$

$$z = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm 2}{2} \nearrow -1 + 0i \quad \searrow -3 + 0i$$

Q4 $z_1 = 5 + i \qquad z_2 = -2 + 3i$

(i) $(z_1)^2 = (5+i)^2 = 25 + 10i + \cancel{i^2}^{-1} = 24 + 10i$

(ii) $|z_1|^2 = 2|z_2|^2$

$|z_1| = \sqrt{5^2+1^2} = \sqrt{26} \Rightarrow |z_1|^2 = 26$

$|z_2| = \sqrt{-2^2+3^2} = \sqrt{13} \Rightarrow |z_2|^2 = 13 \Rightarrow 2|z_2|^2 = 26$

$\therefore |z_1|^2 = 2|z_2|^2$ is True. $26 = 26$.

Q5 $z = -1 + i\sqrt{3}$

(i) $z^2 = (-1 + \sqrt{3}i)^2 = 1 - 2\sqrt{3}i + 3\cancel{i^2}^{\ominus} = -2 - 2\sqrt{3}i$

(ii) $z^2 + pz$ is real.

$-2 - 2\sqrt{3}i + p(-1 + i\sqrt{3}) =$ is real.

$-2 - p - 2\sqrt{3}i + p\sqrt{3}i$ is real

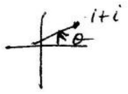
$-2 - p + (p\sqrt{3} - 2\sqrt{3})i$

$\Rightarrow p\sqrt{3} - 2\sqrt{3} = 0$

$p\sqrt{3} = 2\sqrt{3}$

$p = 2.$

Q6 $\quad z = 1 + i$



$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\theta = \tan^{-1} 1 = \pi/4.$

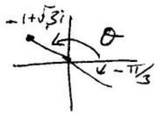
$$z = \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$$

$$z^4 = \left[\sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)\right]^4$$

$$\sqrt{2}^4\left(\cos \pi + i \sin \pi\right)$$

$$4\,(-1 + 0i) \quad = \quad -4 + 0i$$

Q7 $\quad -1 + i\sqrt{3}$



$r = \sqrt{-1^2 + \sqrt{3}^2} = 2$

$\theta = \tan^{-1}(-\sqrt{3}) = -\pi/3$

$\Rightarrow \theta = 2\pi/3$

$$\left(-1 + i\sqrt{3}\right) = 2\left(\cos 2\pi/3 + i \sin 2\pi/3\right)$$

Q8 $\quad 2+3i$ a root of $z^2 - 4z + 13 = 0$

$$(2+3i)^2 - 4\,(2+3i) + 13 = 0$$

$$4 + \cancel{12i} - 9 - 8 - \cancel{12i} + 13 = 0$$

$$17 - 17 = 0.$$

$\Rightarrow$ other root is $2 - 3i$ as all co-effs of $f(z)$ are real.

Q9 $z_1 = 2+3i \qquad z_2 = 1-4i$

$z_1 . z_2 = (2+3i)(1-4i) = 2 - 8i + 3i \overset{+}{-} 12i^2$
$= 14 - 5i$

$|z_1 . z_2| = |14-5i| = \sqrt{14^2 + (-5)^2} = \sqrt{221}$

$|z_1| = |2+3i| = \sqrt{2^2+3^2} = \sqrt{13}$
$|z_2| = |1-4i| = \sqrt{1^2+(-4)^2} = \sqrt{17}$

$\sqrt{13} \times \sqrt{17} = \sqrt{221}$

$\Rightarrow \quad |z_1| . |z_2| = |z_1 . z_2|$

Q10

$\frac{5-5i}{2+i} \times \frac{2-i}{2-i} = \frac{10 - 5i - \cancel{10}i \overset{-}{+} 5i^2}{4+1} = \frac{5-15i}{5}$

$= 1-3i$

Q11 $4i^{13} + 3i^3 \qquad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$

$4(i^{12})(i) + 3i^3$
$4(1)i + 3(-i)$
$4i - 3i \quad = i$

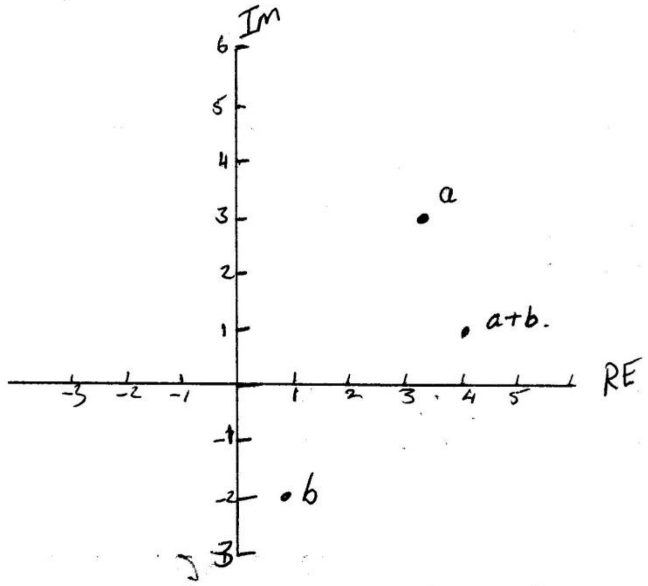
Q12 $z_1 = 2+3i \qquad z_2 = -1+4i$

eqn is $\quad z^2 - \text{Sum}\, z + \text{product} = 0$

Sum: $(2+3i) + (-1+4i) = 1+7i$
Product $(2+3i)(-1+4i) = -2 + 8i - 3i \overset{-}{+} 12i^2 = -14 + 5i$
$\Rightarrow z^2 - (1+7i)z - 14 + 5i$

Q13

$a = 3 + 3i \qquad b = 1 - 2i \qquad a + b = 4 + i$



(i) $a \rightarrow a + b$

$(3+3i) \rightarrow (4+i) \qquad \Rightarrow c = 1 - 2i$

(ii) $b \rightarrow a + b$

$(1-2i) \rightarrow 4 + i \qquad \Rightarrow c = 3 + 3i$

(iii) $a \rightarrow b$

$(3+3i) \rightarrow (1-2i) \qquad \Rightarrow c = -2 - 5i$

Q14

(i) $R \Rightarrow S$ Rotation of $-\pi/2$ radians and stretch by factor $1\frac{1}{2}$.

(ii) $S \Rightarrow T$ Translation of $4 - i$

(iii) ~~let~~ $z = x + iy \in R$. To be $\in S \Rightarrow$ mult by $-i$ and $1\frac{1}{2}$

$zz_1 = (x+iy)(-1\tfrac{1}{2}i)$ Divide both sides by z

$z_1 = -1\tfrac{1}{2}i$

(iv) $zz_1 + z_3 \in T$ To be $\in T \Rightarrow +(4-i)$

$zz_1 + z_3 = (x+iy)(-1\tfrac{1}{2}i) + (4-i) \in T$

$z_3 = 4 - i$

Revision Advanced.

Q1 $z = x + iy$ $\quad 3(z-1) = i(z+1)$

$$\Rightarrow 3(x+iy-1) = i(x+iy+1)$$

$$3x + 3iy - 3 = xi + i^2y + i$$

$$3x - 3 + 3yi = -y + xi + 1i$$

RE $\qquad\qquad$ Im

$$3x - 3 = -y \qquad\qquad 3y = x + 1$$

$$3x + y = 3$$
$$-x + 3y = 1 \quad (\times 3)$$

$$3x + y = 3$$
$$-3x + 9y = 3$$

$$10y = 6$$
$$y = 0\cdot6$$

find x : $\quad -x + 3(0\cdot6) = 1$

$$-x + 1\cdot8 = 1$$
$$-x = -0\cdot8$$
$$x = 0\cdot8$$

Q2

$2+3i$ a root $\Rightarrow$ $2-3i$ is another root.

form Quadratic. $\quad$ Sum : $= 4$

Product : $= 4 + 9 = 13$

$$z^2 - 4z + 13 = 0$$

$$\begin{array}{r|l} & 2z - 1 \\ z^2 - 4z + 13 & 2z^3 - 9z^2 + 30z - 13 = 0 \\ & -2z^3 + 8z^2 - 26z \\ & \quad -z^2 + 4z - 13 \\ & \quad -z^2 + 4z - 13 \end{array}$$

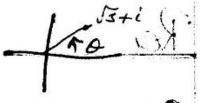
$2z - 1$ is a factor $\quad \Rightarrow \quad 2z - 1 = 0$

$$2z = 1$$

$z = 1/2$ is 3^{rd} root.

Q3 $\sqrt{3}+i$ $\quad r=\sqrt{(\sqrt{3})^2+(1)^2}=2$

$\theta=\tan^{-1}\frac{1}{\sqrt{3}}=\pi/6$



$$\Rightarrow \sqrt{3}+i = 2\left(\cos \pi/6 + i\sin \pi/6\right)$$

$$\therefore (\sqrt{3}+i)^{11} = \left[2\left(\cos \pi/6 + i\sin \pi/6\right)\right]^{11}$$

$$= 2^{11}\left(\cos (11)\pi/6 + i\sin (11)\pi/6\right)$$

$$= 2^{11}\left(\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right)$$

$$= 2^{10}\left(\sqrt{3}-i\right)$$

Q4 $z^2+pz+q=0$ $\quad 1+i$ and $4+3i$

$$p = -(\text{Sum}) = -\left[(1+i)+(4+3i)\right] = -(5+4i)$$

$$\boxed{p = -5-4i}$$

$$q = \text{Product} \quad (1+i)(4+3i) = 4+3i+4i \overset{-}{+} 3\cancel{i^2}$$

$$= 1+7i$$

$$\boxed{q = 1+7i}$$

Q5 $w_1 = \frac{-1}{2} + \frac{\sqrt{3}}{2} i$

$w_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^2 = \frac{1}{4} - \frac{\sqrt{3}}{2} i \overset{\ominus}{+} \frac{3}{4} i^2$ $\quad 2\left(\frac{-\sqrt{3}}{4}\right) = \frac{-\sqrt{3}}{2}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$w_1 + w_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -1 + 0i$
$= -1.$

Q6 $p = 2\left(\cos\frac{\pi}{3} + i \sin \pi/3\right)$

$\bar{p} = (2)\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\pi/3\right)\right)$

$p\bar{p} = 4\left(\cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right) + i \sin\left(\pi/3 - \pi/3\right)\right)$

$= 4(\cos 0 + i \sin 0)$

$= 4(1 + 0i) = 4 + 0i = 4.$ = real No.

Q7 $\frac{(1+2i)^2}{1-i} = \frac{1 + 4i \overset{\ominus}{+} \cancel{4} i^2}{1-i} = \frac{\cancel{-3}+4i}{1-i}$

$\frac{-3+4i}{1-i} \times \frac{1+i}{1+i} = \frac{-3-3i+4i \overset{\ominus}{+} 4i^2}{1+1} = \frac{-7+i}{2}$

$= -\frac{7}{2} + \frac{1}{2} i$

Q8 $\dfrac{-3+i}{1+ki} \times \dfrac{1-ki}{1-ki} = \dfrac{-3+3ki+i-ki^2}{1+k^2}$ 20

$= \dfrac{-3+k}{1+k^2} + \dfrac{3k+1}{1+k^2}i$

Real part $= -3$

$\Rightarrow \dfrac{-3+k}{1+k^2} = -3$

$\cancel{-3}+k = \cancel{-3}-3k^2$

$3k^2+k=0$

$k(3k+1)=0$

$\cancel{k=0}$ $\quad 3k+1=0$

$3k=-1$

$k=-1/3$

Q9 $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)^2$

$= \left(\cos \pi/3 + i\sin\pi/3\right)\left(\cos\frac{\pi}{6}+i\sin\pi/6\right)$

$= \cos\left(\pi/3+\pi/6\right) + i\sin\left(\pi/3+\pi/6\right)$

$= \cos\frac{\pi}{2} + i\sin\pi/2$

$= 0+1i$

Q10 $1+2i$ Root of $z^2-(3+3i)z+5i=0$

$(1+2i)^2-(3+3i)(1+2i)+5i=0$

$1+4i+4i^2-3-6i-3i-6i^2+5i=0$

$0=0$ True.

It is not $\bar{z}$ as coeffs are not $\in R$.

$\Rightarrow$ Sum of roots $= 3+3i$

$(1+2i)+(a+bi) = 3+3i$

$\Rightarrow 2^{nd}$ root is $2+i$

Q11 $\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^4$

$= \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\left(\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right) + i \sin\left(\frac{2\pi}{3} + \frac{8\pi}{3}\right)$

$= \cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

Q12 $z^3 + z^2 + 4z + p = 0$ $\quad 1-3i$ is a root.

To find p.

$(1-3i)^3 + (1-3i)^2 + 4(1-3i) + p = 0.$

$(1)^3 + 3(-3i) + 3(-3i)^2 + (-3i)^3 + 1 - 6i + 9i^2 + 4 - 12i + p = 0$

$1 - 9i - 27i^2 - 27i + 1 - 6i - 9 + 4 - 12i + P = 0$

$-30 + p = 0$

$p = 30$

2^{nd} Root $= 1+3i$ as all coeffs are real.

Sum $= (1+3i) + (1-3i) = 2$

Product $= (1+3i)(1-3i) = 1+9 = 10$

Quadratic $= z^2 - 2z + 10.$

$$
\begin{array}{r}
z + 3 \\
z^2 - 2z + 10 \,\overline{)\, z^3 + z^2 + 4z + 30} \\
-\left(z^3 - 2z^2 + 10z\right) \\
\hline
3z^2 - 6z + 30 \\
3z^2 - 6z + 30 \\
\hline
\end{array}
$$

$z + 3 = 0 \qquad z = -3$ is the 3^{rd} root

$\therefore$ Other 2 roots are $1+3i$ and -3.

Q13 $x + iy = \sqrt{8-6i}$ sq both sides. 11 (2)

$x^2 + 2xyi - y^2 = 8 - 6i$

RE

$x^2 - y^2 = 8$

$x^2 - \left(\frac{-3}{x}\right)^2 = 8$

$x^2 - \frac{9}{x^2} = 8$

$x^4 - 9 = 8x^2$

$x^4 - 8x^2 - 9 = 0$

$(x^2 - 9)(x^2 + 1) = 0$

$x^2 = 9$ $\quad$ $x^2 = -1$

$x = \pm 3$ $\quad$ ~~$x = \sqrt{-1} = i$~~ we assume x and y are real

IM

$2xy = -6$

$y = -3/x$

find y: at $x = 3$ $\quad y = -3/x$

$y = -3/3 = -1$

at $x = -3$ $\quad y = -3/-3 = 1.$

$\Rightarrow x = 3,\ y = -1$ $\quad$ or $\quad x = -3,\ y = 1$

Q14 $\quad z^4 - 2z^3 + 7z^2 - 4z + 10 = 0 \quad ti$ is a sol.

$\Rightarrow (ti)^4 - 2(ti)^3 + 7(ti)^2 - 4(ti) + 10 = 0$

$t^4 + 2t^3 i - 7t^2 - 4ti + 10 = 0$

RE	IM
$t^4 - 7t^2 + 10 = 0$	$2t^3 - 4t = 0$
$(t^2 - 5)(t^2 - 2) = 0$	$2t(t^2 - 2) = 0$
$t^2 = 5 \quad t^2 = 2$	$2t = 0 \quad t^2 = 2$
$t = \pm\sqrt{5} \quad t = \pm\sqrt{2}$	$t = 0 \quad t = \pm\sqrt{2}$

The only values common are $t = \pm\sqrt{2}$.

$\Rightarrow t = \sqrt{2} \Rightarrow \sqrt{2}i$ is a Sol z_1

$t = -\sqrt{2} \Rightarrow -\sqrt{2}i$ is a Sol z_2.

form Quadratic : Sum: $\sqrt{2}i + (-\sqrt{2}i) = 0$

Prod: $(\sqrt{2}i)(-\sqrt{2}i) = 2$

$z^2 - 0z + 2 = 0 \quad = \quad z^2 + 2 = 0$.

$$\begin{array}{r}
z^2 - 2z + 5 \\
z^2 + 2 \,\big)\, \overline{z^4 - 2z^3 + 7z^2 - 4z + 10} \\
-\; z^4 \quad\;\; -\; 2z^2 \qquad\qquad\quad \\
\hline
-2z^3 + 5z^2 - 4z + 10 \\
+\; -2z^3 \quad\;\; +\; 4z \qquad\quad \\
\hline
5z^2 + 10 \\
5z^2 + 10 \\
\hline
\end{array}$$

$z^2 - 2z + 5 = 0$ need to use formula

$z = \dfrac{2 \pm \sqrt{4 - 4(5)}}{2} = \dfrac{2 \pm 4i}{2} = 1 \pm 2i$

$\therefore$ Sols are: $\sqrt{2}i, \; -\sqrt{2}i, \; 1+2i$ and $1-2i$.

Q15 $\quad z = a + bi \quad \Rightarrow \bar{z} = a - bi$

$$z\bar{z} - 2iz = 7 - 4i$$
$$(a+bi)(a-bi) - 2(a+bi)i = 7 - 4i$$
$$a^2 + b^2 - 2ai + 2b = 7 - 4i$$

RE $\qquad\qquad$ Im

$a^2 + b^2 + 2b = 7 \qquad -2a = -4$

$\qquad\qquad\qquad\qquad a = 2$

$$(2)^2 + b^2 + 2b = 7$$
$$b^2 + 2b - 3 = 0$$
$$(b+3)(b-1) = 0$$
$$b = -3 \qquad b = 1$$

$z = a + bi \Rightarrow z = 2 - 3i$ or $2 + i$

Q16 $\quad (p + iq)^2 = 15 - 8i$

$$p^2 + 2pqi - q^2 = 15 - 8i$$

RE $\qquad\qquad$ Im

$p^2 - q^2 = 15 \qquad 2pq = -8$

$\qquad\qquad\qquad q = -4/p$

$$p^2 - \left(\frac{-4}{p}\right)^2 = 15$$
$$p^2 - \frac{16}{p^2} = 15$$
$$p^4 - 15p^2 - 16 = 0$$
$$(p^2 - 16)(p^2 + 1) = 0$$

$p^2 = 16 \qquad p^2 = -1$

$p = \pm 4 \qquad \cancel{p = \pm\sqrt{-1} = \pm i}$ but p is real.

find q: When $p = 4$: $\quad q = -4/4 = -1 \quad \boxed{4 - i}$

When $p = -4 \quad q = -4/-4 = 1 \quad \boxed{-4 + i}$

$\Rightarrow (4 - i)^2 = 15 - 8i$ and $(-4+i)^2 = 15 - 8i$

Hence solve $(1+i)z^2+(-2+3i)z-3+2i=0$

Use formula to solve

$$z=\frac{-(-2+3i)\pm\sqrt{(-2+3i)^2-4(1+i)(-3+2i)}}{2(1+i)}$$

$$z=\frac{2-3i\pm\sqrt{4-12i-9+20+4i}}{2+2i}$$

$$=\frac{2-3i\pm\sqrt{15-8i}}{2+2i}$$

$(-4-4i)(-3+2i) = 12-8i+12i+8 = 20+4i$

But $\sqrt{15-8i} = 4-i$ or $-4+i$ $\boxed{\text{ie } \pm(4-i)}$

$$\Rightarrow \frac{(2-3i)\pm(4-i)}{2+2i}$$

$$\frac{2-3i+4-i}{2+2i}$$

$$\frac{6-4i}{2+2i}\times\frac{2-2i}{2-2i}$$

$$=\frac{12-12i-8i-8}{4+4}$$

$$=\frac{4-20i}{8}$$

$$=\frac{1}{2}-\frac{5}{2}i$$

$$\frac{2-3i-4+i}{2+2i}$$

$$\frac{-2-2i}{2+2i}=-\frac{(2+2i)}{2+2i}=-1$$

$\Rightarrow$ Sols are $\frac{1}{2}-\frac{5}{2}i$ and -1

Revision Exercise [Extended Response Q's]

Q1 $p = 3(\cos\pi/6 + i\sin\pi/6)$ $q = 2 - 2i\sqrt{3}$

(i) pq: first express p in the form $a+bi$

$$p = 3(\cos\pi/6 + i\sin\pi/6)$$
$$= 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i.$$

$$\Rightarrow pq = \left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right)\left(2 - 2\sqrt{3}i\right)$$
$$= 3\sqrt{3} - 9i + 3i \oplus 3\sqrt{3}\cancel{i}$$
$$= 6\sqrt{3} - 6i$$

(ii) $$|p| = \left|\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right| = \sqrt{\left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$$
$$= \sqrt{\frac{27}{4} + \frac{9}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$$|q| = \sqrt{2 - 2\sqrt{3}i} = \sqrt{(2)^2 + (-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

$$|pq| = |6\sqrt{3} - 6i| = \sqrt{(6\sqrt{3})^2 + (-6)^2} = \sqrt{108+36} = \sqrt{144} = 12$$

$$|p+q| = \left|\left(\frac{3\sqrt{3}}{2} + \frac{3}{2}i\right) + \left(2 - 2\sqrt{3}i\right)\right| = \left|\frac{4+3\sqrt{3}}{2} + \frac{3-4\sqrt{3}}{2}\right|$$

$$= \sqrt{\left(\frac{4+3\sqrt{3}}{2}\right)^2 + \left(\frac{3-4\sqrt{3}}{2}\right)^2} = \sqrt{\frac{43+\cancel{24\sqrt{3}}}{4} + \frac{57-\cancel{24\sqrt{3}}}{4}}$$

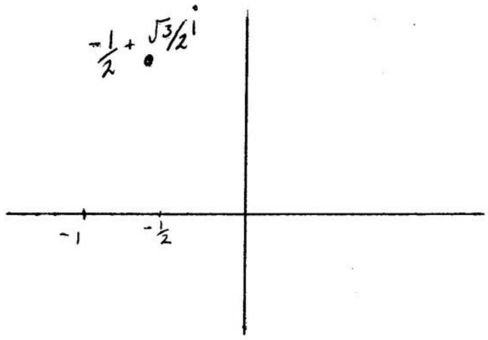
$$= \sqrt{\frac{100}{4}} = \sqrt{25} = 5$$

Q2 $\quad z = \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$

(i) $\quad \dfrac{1+\sqrt{3}i}{1-\sqrt{3}i} \times \dfrac{1+\sqrt{3}i}{1+\sqrt{3}i} = \dfrac{1+\sqrt{3}i+\sqrt{3}i+3i^2}{1+3} = \dfrac{-2+2\sqrt{3}i}{4}$

$= -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

(ii)



(iii) $\quad -\frac{1}{2} + \sqrt{3}/2\, i \qquad r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$\theta = \tan^{-1}\dfrac{\sqrt{3}/2}{-\frac{1}{2}} = \tan^{-1} -\sqrt{3} = -\dfrac{\pi}{3}$

$\Rightarrow \theta = 2\pi/3$

$\therefore \; -\frac{1}{2} + \sqrt{3}/2\, i = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(iv) $\quad z^3 = \left[1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^3 = 1\left(\cos 2\pi + i\sin 2\pi\right)$

$= 1 + 0i = 1.$ True.

Q3 $z = (1+3i)(p+qi)$

(i) $z = p + qi + 3pi + 3qi^2$

$= (p-3q) + (3p+q)i$

argument of $z = \pi/4$. $\Rightarrow \theta = \pi/4$

$$\tan^{-1}\left(\frac{3p+q}{p-3q}\right) = \frac{\pi}{4}$$

$$\frac{3p+q}{p-3q} = \tan(\pi/4)$$

$$\frac{3p+q}{p-3q} = 1$$

$$3p + q = p - 3q$$

$$2p + 4q = 0 \quad (\div 2)$$

$$p + 2q = 0 \qquad \text{QED}$$

(ii) $|z| = 10\sqrt{2}$

$\Rightarrow |(p-3q) + (3p+q)i| = 10\sqrt{2}$

$$\sqrt{(p-3q)^2 + (3p+q)^2} = 10\sqrt{2}$$

$$\sqrt{p^2 - \cancel{6pq} + 9q^2 + 9p^2 + \cancel{6pq} + q^2} = 10\sqrt{2}$$

$\sqrt{10p^2 + 10q^2} = 10\sqrt{2}$ sq both sides

$$10p^2 + 10q^2 = 200$$

$$p^2 + q^2 = 20.$$

But from above $p+2q = 0$

$p^2+q^2=20$ $\qquad$ $p+2q=0$

$\qquad\qquad\qquad\qquad p=-2q$

$(-2q)^2+q^2=20$

$5q^2=20$

$q^2=4$

$q=\pm 2$

find P: $\qquad P=-2q$.

at $q=2$. $\quad P=-2(2)=-4$ $\quad$ <u>but $P>0$</u>

at $q=-2$ $\quad p=-2(-2)=4$.

$\Rightarrow$ ans $\quad P=4$, $q=-2$

<u>Q4</u> $\quad z_1 = 3\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ $\quad z_2=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

(i) $|z_1 z_2| = \left|3\left(\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)+i\sin\left(\pi/6+\pi/4\right)\right)\right|$

$= \left|3\left(\cos\frac{5\pi}{12}+i\sin 5\pi/12\right)\right| = 3$ as $r=3$

(Rem: r = The modulus of the Number) $\quad |z_1z_2|$ = Dis from origin.

(ii) arg of $(z_1z_2) = \frac{5\pi}{12}$

(iii) $|z_1|^2$ $\qquad |z_1|=3$. $\Rightarrow |z_1|^2=9$.

(iv) $|z_2|^2$ $\qquad |z_2|=1$ $\Rightarrow |z_2|^2=1$

(v) $\arg(z_1)^2$: $(z_1)^2 = \left[3\left(\cos\pi/6+i\sin\pi/6\right)\right]^2$

$= 9\left(\cos\frac{\pi}{3}+i\sin\pi/3\right) \Rightarrow \theta=\frac{\pi}{3}$

(vi) $\arg(z_2)^2$: $(z_2)^2 = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2$

$= \cos\pi/2 + i\sin\pi/2$

$\Rightarrow \theta = \pi/2$

(vii) (a) $|zw| = |z||w|$

Let $z = a+bi$ $\quad w = x+yi$

$zw = (a+bi)(x+yi) = ax + ayi + bxi - by$

$(ax-by) + (ay+bx)i$

$|z| = \sqrt{a^2+b^2}$

$|w| = \sqrt{x^2+y^2}$

$|z||w| = \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right) = \sqrt{a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2}$

$|zw| = |(ax-by) + (ay+bx)i| = \sqrt{(ax-by)^2 + (ay+bx)^2}$

$= \sqrt{a^2x^2 \cancel{-2abxy} + b^2y^2 + a^2y^2 + \cancel{2abxy} + b^2x^2}$

$= \sqrt{a^2x^2 + b^2y^2 + a^2y^2 + b^2x^2}$

$\Rightarrow$ True $|zw| = |z|.|w|$

(b) $\arg(zw) = \arg(z) + \arg(w)$

$\arg z = \tan^{-1}\frac{b}{a}$ $\qquad$ $\arg w = \tan^{-1}\frac{y}{x}$

$\arg(zw) = \tan^{-1}\left(\frac{ay+bx}{ax-by}\right)$

$\arg z + \arg w = \tan^{-1}\left(\frac{b}{a}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Let $m = \tan^{-1}\frac{b}{a}$ and $n = \tan^{-1}\frac{y}{x}$

$\Rightarrow \tan m = \frac{b}{a}$ $\qquad$ $\Rightarrow \tan n = \frac{y}{x}$.

from Tables. $\tan(m+n) = \dfrac{\tan m + \tan n}{1 - \tan m \tan n}$

$$= \frac{\frac{b}{a} + \frac{y}{x}}{1 - \left(\frac{b}{a}\right)\left(\frac{y}{x}\right)} = \frac{\frac{bx+ya}{ax}}{1 - \frac{by}{ax}}$$

$$= \frac{\frac{bx+ya}{ax}}{\frac{ax-by}{ax}} = \frac{bx+ya}{\cancel{ax}} \times \frac{\cancel{ax}}{ax-by} = \frac{bx+ya}{ax-by}.$$

$= \arg(zw)$

$\Rightarrow$ True $\arg(z) + \arg(w) = \arg(zw)$.

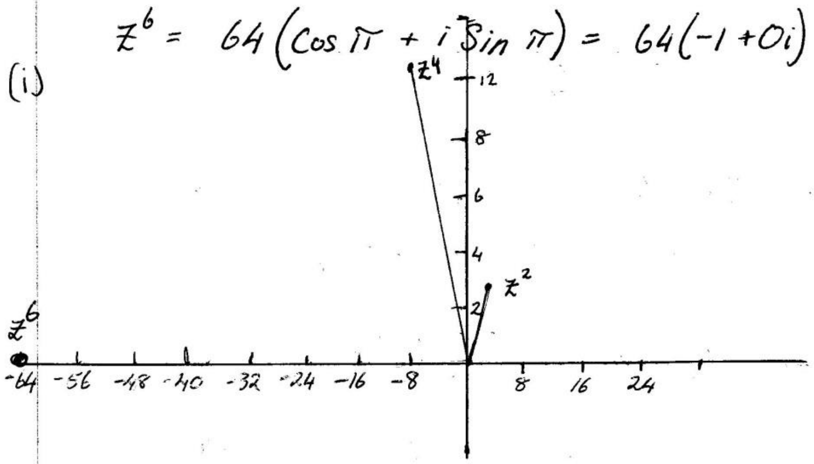
Q5 $z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$z^2 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2 + 2\sqrt{3}i$$

$$z^4 = 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -8+8\sqrt{3}i$$

$$z^6 = 64\left(\cos\pi + i\sin\pi\right) = 64(-1+0i) = -64+0i$$

(i)

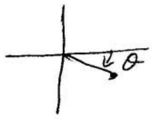


(ii) The transformation is a rotation of $\pi/3$ and stretching of factor 4.

Q6 $$\frac{\sqrt{3}+i}{1+i\sqrt{3}} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{\sqrt{3}-3i+i\overset{+}{-}\sqrt{3}\cancel{i^2}}{1+3} = \frac{2\sqrt{3}-2i}{4}$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$r = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1} = 1$$



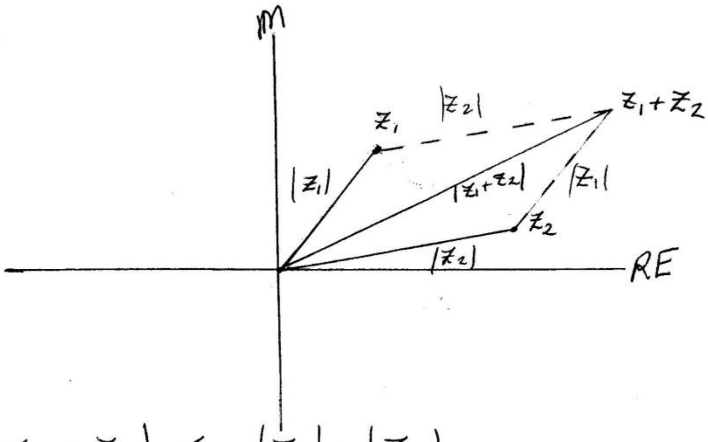
$$\theta = \tan^{-1}\left(\frac{-1/2}{\sqrt{3}/2}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

$$= 1\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)$$

$$z^6 = 1^6\left(\cos -\pi + i\sin -\pi\right)$$

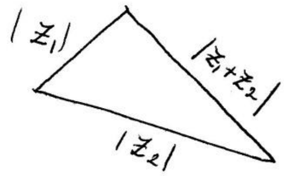
$$(-1 + 0i) = -1$$

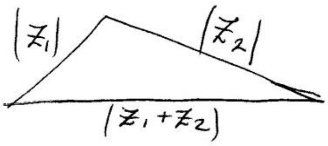
Q7



$|z_1 + z_2| \leq |z_1| + |z_2|$

Diagonals in a parallelogram are equal. OR. opp sides are equal





$|z_1+z_2| < |z_1|+|z_2|$

for $|z_1+z_2| = |z_1|+|z_2|$ z_1 and z_2 must lie on the <u>same</u> line through the origin.

Q8 $zw = 1$

(i) $z = a + bi \qquad w = c + di$

$$(a+bi)(c+di) = 1$$
$$ac + adi + bci - bd = 1$$

$$\underline{\underline{RE}} \qquad \qquad \underline{\underline{IM}}$$
$$ac - bd = 1 \qquad \qquad ad + bc = 0$$

(ii) To express a in terms of c and d $\Rightarrow$ eliminate b.

$$ad + bc = 0$$
$$bc = -ad$$
$$b = \frac{-ad}{c}$$

$$ac - bd = 1$$
$$ac - \left(\frac{-ad}{c}\right)(d) = 1$$
$$ac + \frac{ad^2}{c} = 1 \quad (\times c)$$
$$ac^2 + ad^2 = c$$
$$a(c^2+d^2) = c$$
$$\boxed{a = \frac{c}{c^2+d^2}}$$

To express b in terms of c and d $\Rightarrow$ eliminate a.

$$ad + bc = 0$$
$$ad = -bc$$
$$a = \frac{-bc}{d}.$$

$$ac - bd = 1$$
$$\left(\frac{-bc}{d}\right)(c) - bd = 1$$
$$\frac{-bc^2}{d} - bd = 1 \quad (\times -d)$$
$$bc^2 + bd^2 = -d$$
$$b(c^2+d^2) = -d$$
$$\boxed{b = \frac{-d}{c^2+d^2}}$$

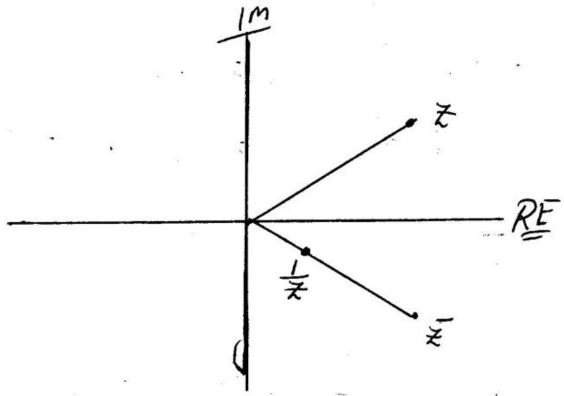
(iii) $z = a+bi \qquad \bar{z} = a-bi$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$\frac{1}{z} = \frac{1}{a+bi} \times \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}$$

$$\frac{\bar{z}}{|z|^2} = \frac{a-bi}{\left(\sqrt{a^2+b^2}\right)^2} = \frac{a-bi}{a^2+b^2}$$

(iv)



(v) If collinear $\Rightarrow$ same slope

$\bar{z} = a - bi$

Slope of $\bar{z} = \frac{-b}{a}$

$$\frac{1}{z} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

$$\text{Slope of } \frac{1}{z} = \frac{\frac{-b}{a^2+b^2}}{\frac{a}{a^2+b^2}} = \frac{-b}{\cancel{a^2+b^2}} \times \frac{\cancel{a^2+b^2}}{a} = \frac{-b}{a}$$

$\Rightarrow$ Same slopes $\Rightarrow$ are collinear.

Q9 (i) Let $z = a+bi \quad w = c+id.$

Prove : $\overline{z+w} = \bar{z} + \bar{w}$

$$\overline{(a+bi)+(c+di)} = a-bi + c-di$$

$$\overline{(a+c)+(b+d)i} = a+c - bi - di$$

$$(a+c)-(b+d)i = (a+c) - (b+d)i \quad \text{True}$$

QED

(ii) Prove : $\overline{z-w} = \bar{z} - \bar{w}$

$$\overline{(a+bi)-(c+di)} = (a-bi) - (c-di)$$

$$\overline{(a-c)+(b-d)i} = a-bi - c + di$$

$$(a-c)-(b-d)i = (a-c) - (b-d)i \quad \text{True}$$

QED.

(iii) Prove: $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$$\left(\frac{z}{w}\right) = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{ac-adi+bci+bd}{c^2+d^2}$$

$$= \frac{(ac+bd) - (ad-bc)i}{c^2+d^2}$$

$$\Rightarrow \overline{\left(\frac{z}{w}\right)} = \frac{(ac+bd)+(ad-bc)i}{c^2+d^2}$$

$$\frac{\bar{z}}{\bar{w}} = \frac{a-bi}{c-di} \times \frac{c+di}{c+di} = \frac{ac+adi-bci+bd}{c^2+d^2} = \frac{(ac+bd)+(ad-}{c^2+d^2}$$

$$\therefore \text{True} \ \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$$

Q.E.D.

(iv) Prove $\overline{(zw)} = \bar{z}.\bar{w}$

$$\overline{(a+bi)(c+di)} = (a-bi)(c-di)$$

$$\overline{ac+adi+bci-bd} = ac-adi-bci-bd$$

$$\overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$$

$$(ac-bd)-(ad+bc)i = (ac-bd)-(ad+bc)i \text{ True.}$$

QED.

Q10 (a) $w = -1+\sqrt{3}i$

(i) $r = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$ $\quad\theta = \operatorname{Tan}^{-1}\left(-\frac{\sqrt{3}}{1}\right) = \frac{-\pi}{3}$

$\Rightarrow \theta = \frac{2\pi}{3}$

$$-1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

(ii) $z^2 = -1+\sqrt{3}i$

$$z = (-1+\sqrt{3}i)^{\frac{1}{2}}$$

$$= \left[2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^{\frac{1}{2}} \quad \text{must use general form.}$$

$$= \left[2\left(\cos\frac{2\pi}{3}+2n\pi + i\sin\frac{2\pi}{3}+2n\pi\right]^{\frac{1}{2}}$$

$$= 2^{\frac{1}{2}}\left[\cos\left(\frac{\frac{2\pi}{3}+2n\pi}{2}\right) + i\sin\left(\frac{\frac{2\pi}{3}+2n\pi}{2}\right)\right]$$

let $n=0$ $\quad 2^{\frac{1}{2}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2^{\frac{1}{2}}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

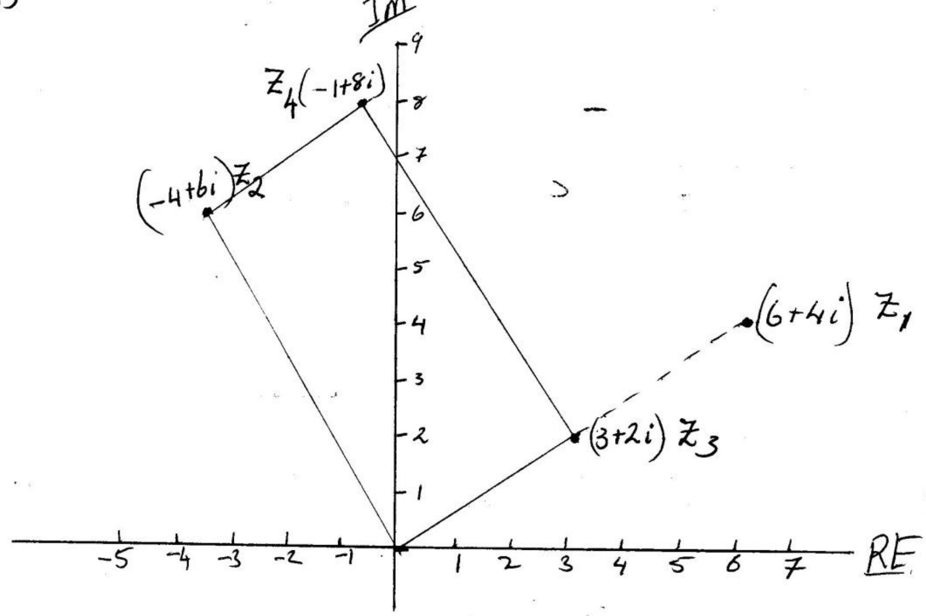
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$$

let $n=1$ $\quad 2^{\frac{1}{2}}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{2} + i\sin\frac{\frac{2\pi}{3}+2\pi}{2}\right)$

$$= 2^{\frac{1}{2}}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 2^{\frac{1}{2}}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$$

(b) (i) $z_1, \quad z_2 = iz_1, \quad z_3 = kz_1, \quad z_4 = z_2 + z_3$



(ii) $k = \frac{1}{2}$

(iii) $z_2 + z_3 = z_4$ forms a //gram.
also $z_3 = kz_1 \Rightarrow z_3$ and z_1 are collinear.

Q11 (a) (i) $1-i$

$r = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}(-1) = -\pi/4$

$1-i = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\pi/4\right)\right)$

(ii) $(1-i)^9 = \left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin -\pi/4\right)\right]^9$

$= \sqrt{2}^9\left(\cos -\frac{9\pi}{4} + i\sin -\frac{9\pi}{4}\right)$

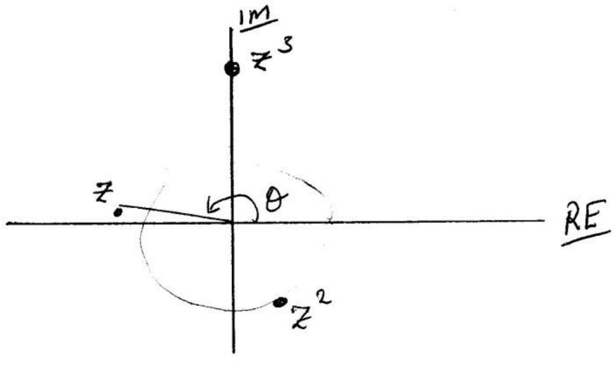
$= 2^{9/2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$16 - 16i$ (Cal).

(b)
(i) $|z| > 1$

$\Rightarrow |z^2| > 2$

$|z^3| > |z^2|$ as mult by z The complex N° rotates and stretches.



(ii) θ z^3 lies on im axis

$z^3 = \left[r\left(\cos\theta + i\sin\theta\right)\right]^3$

$= r^3\left(\cos 3\theta + i\sin 3\theta\right)$

$3\theta = 2\pi + \frac{\pi}{2}$

$3\theta = \frac{5\pi}{2} \Rightarrow \theta = \frac{5\pi}{6}$.

(iii) $|z| > 1$ results in stretching away from the origin.

$\boxed{\text{Pg } 109}$

(c) $z = a + ai \quad a > 1$

(i) $z^2 = (a+ai)(a+ai)$

$= a^2 + 2a^2 i \cancel{- a^2}$

$= 2a^2 i \quad = 0 + 2a^2 i$

$z^4 = (2a^2 i)(2a^2 i)$

$= -4a^4 \quad = -4a^4 + 0i$

$z^6 = (z^4)(z^2)$

$= (2a^2 i)(-4a^4)$

$= -8a^6 i \quad = 0 - 8a^6 i$

(ii) Points spiral away from the origin and lie on the RE and Im axis.

Q12 (i) $z = \cos\theta + i\sin\theta$.

$z^k = (\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

(ii) show $\frac{1}{z^k} = \cos k\theta - i\sin k\theta$

$$\frac{1}{\cos k\theta + i\sin k\theta} \times \frac{\cos k\theta - i\sin k\theta}{\cos k\theta - i\sin k\theta} = \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta + \sin^2 k\theta}$$

$\boxed{\text{but } \cos^2\theta + \sin^2\theta = 1}$ $= \cos k\theta - i\sin k\theta$ QED.

(iii) $z^k = \cos k\theta + i\sin k\theta$ (A)

$z^{-k} = \cos k\theta - i\sin k\theta$ (B)

Add (A)+(B)

$z^k + z^{-k} = 2\cos k\theta$

$$\boxed{\frac{z^k + z^{-k}}{2} = \cos k\theta}$$

$z^k = \cos k\theta + i\sin k\theta$

$(-)\ z^{-k} = \cos k\theta - i\sin k\theta$ subtract

$z^k - z^{-k} = 2i\sin k\theta$

$$\boxed{\frac{z^k - z^{-k}}{2i} = \sin k\theta}$$

(iv) Show $\cos^2\theta \sin^2\theta = \frac{-1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$

Tables $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \frac{\sin 2\theta}{2} = \sin\theta\cos\theta$

$\therefore \sin^2\theta\cos^2\theta = \left(\frac{\sin 2\theta}{2}\right)^2$

from Previous part $\sin k\theta = \frac{z^k - z^{-k}}{2i}$

In this Case $k=2 \Rightarrow \sin 2\theta = \frac{z^2 - z^{-2}}{2i}$

Hence: $\sin^2\theta\cos^2\theta = \left(\frac{\sin 2\theta}{2}\right)^2 = \left(\frac{\frac{z^2 - z^{-2}}{2i}}{2}\right)^2$

$= \left(\frac{z^2 - z^{-2}}{2i} \times \frac{1}{2}\right)^2 = \left(\frac{z^2 - z^{-2}}{4i}\right)^2$

$= \left(\frac{1}{4i}\right)^2 \left(z^2 - z^{-2}\right)^2$

$\frac{-1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$ Q.E.D.

(v) Show $\cos^2\theta \sin^2\theta = a + b\cos 4\theta$

$$\cos^2\theta \sin^2\theta = -\frac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2$$

$$= -\tfrac{1}{16}\left(z^4 - 2 + \frac{1}{z^4}\right)$$

$$= -\tfrac{1}{16}\left(z^4 + \frac{1}{z^4} - 2\right)$$

Need a $\cos 4\theta$ → from before $\cos k\theta = \dfrac{z^k + z^{-k}}{2}$.

In this case $k = 4$. $\Rightarrow \cos 4\theta = \dfrac{z^4 + z^{-4}}{2}$

$$\Rightarrow 2\cos 4\theta = z^4 + z^{-4}.$$

$$\therefore \cos^2\theta \sin^2\theta = -\frac{1}{16}\left(2\cos 4\theta - 2\right)$$

$$= -\frac{1}{8}\cos 4\theta + \frac{1}{8}.$$

$\Rightarrow a = \frac{1}{8}$ and $b = -\tfrac{1}{8}$.