Revision Exercise 2. Advanced.

1. $y = \sin x - \cos x$ at $x = \frac{\pi}{2}$.

$$\frac{dy}{dx} = \cos x - (-\sin x)$$
$$= \cos x + \sin x$$

at $x = \frac{\pi}{2}$ $\frac{dy}{dx} = \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)$
$$= 0 + 1 = 1$$

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2. Slope at $x = \frac{\pi}{2}$ $\Rightarrow \frac{dy}{dx}$ $\quad y = x^2 \sin x$ (product rule)

$$\frac{dy}{dx} = x^2 \cos x + \sin x . 2x$$

at $x = \frac{\pi}{2}$ $\Rightarrow \left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)$
$$= \frac{\pi^2}{4}(0) + \pi(1)$$
$$= 0 + \pi = \pi.$$

Slope of Tangent at $x = \frac{\pi}{2} = \pi$

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3. Slope $= 3$ $\Rightarrow \frac{dy}{dx} = 3$.

$y = x^2 + \ln x$,

$\frac{dy}{dx} = 2x + \frac{1}{x}$

$2x + \frac{1}{x} = 3$

$\frac{2x^2+1}{x} = 3$

$2x^2 + 1 = 3x$

$2x^2 - 3x + 1 = 0$

$(2x - 1)(x - 1) = 0$

$2x = 1$ $\quad x = 1$

$x = 1/2$

find y:

at $x = \frac{1}{2}$, $y = \left(\frac{1}{2}\right)^2 + \ln\frac{1}{2} = \frac{1}{4} + \ln 1 - \ln 2$
$= \frac{1}{4} + 0 - \ln 2$
$= \frac{1}{4} - \ln 2$

$\left(\frac{1}{2}, \frac{1}{4} - \ln 2\right)$

at $x = 1$, $y = (1)^2 + \ln(1)$
$= 1 + 0$
$= 1$

$(1, 1)$

$\underline{4}$ $y = x - 1 + \frac{1}{x-1}$ find values for which $\frac{dy}{dx} = 0$.

$$y = \frac{x(x-1) - 1(x-1) + 1}{x-1}$$

$$y = \frac{x^2 - x - x + 1 + 1}{x-1}$$

$$y = \frac{x^2 - 2x + 2}{x-1}$$

$$\frac{dy}{dx} = \frac{(x-1)(2x-2) - (x^2-2x+2)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - 2x + 2 - x^2 + 2x - 2}{x^2 - 2x + 1}$$

$$= \frac{x^2 - 2x}{x^2 - 2x + 1} = 0$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \quad \underline{x = 2}$$

$\underline{Q5}$ (i) $y = \ln 3x^4$

$$\frac{dy}{dx} = \frac{1}{3x^4} \cdot 12x^3$$

$$= \frac{12x^3}{3x^4} = \frac{4}{x}$$

(ii) $y = \ln\left(\frac{3}{\sqrt{x}}\right)$

$$y = \ln 3 - \ln\sqrt{x}$$

$$\frac{dy}{dx} = \cancel{\frac{1}{3}(0)} - \frac{1}{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{-1}{x^{\frac{1}{2}}} \cdot \frac{1}{2x^{\frac{1}{2}}}$$

$$= -\frac{1}{2x^1}$$

$$= -\frac{1}{2}x$$

6/ $y = e^{nx}$

$\frac{dy}{dx} = e^{nx}.(n)$

$= ne^{nx}$

$\frac{d^2y}{dx^2} = ne^{nx}.(n)$

$= n^2e^{nx}$.

$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

$n^2e^{nx} - 5ne^{nx} + 6e^{nx} = 0$

$(\div e^{nx})$ $n^2 - 5n + 6 = 0$

$(n - 3)(n - 2) = 0$

$n = 3$ and $n = 2$

7/ $y = x^3 - 3x^2 - 5x + 10$

parallel to $y = 4x - 7$

$\Rightarrow$ Slope $= 4$.

$\frac{dy}{dx} = 3x^2 - 6x - 5$

$3x^2 - 6x - 5 = 4$.

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ $\quad x = -1$

find y co-ords.

at $x = 3$, $y = (3)^3 - 3(3)^2 - 5(3) + 10$

$y = -5$

$(3, -5)$

at $x = -1$, $y = (-1)^3 - 3(-1)^2 - 5(-1) + 10$

$y = 11$

$(-1, 11)$.

8. $y = a\sqrt{x} - 5$ slope $= 2$ at $(4, b)$

$y = ax^{\frac{1}{2}} - 5$

$\frac{dy}{dx} = \frac{1}{2} a x^{-\frac{1}{2}}$

$= \frac{a}{2\sqrt{x}} = 2$ at $x = 4$

$\frac{a}{2\sqrt{4}} = 2$

$\frac{a}{4} = 2$

$a = 8$

$y = a\sqrt{x} - 5$ at $(4, b)$, $a = 8$

$b = 8\sqrt{4} - 5$

$b = 16 - 5$

$b = 11$

9.

(i) $V = 80(30-t)^3$

A is where cuts V axis $\Rightarrow t = 0$

$V = 80(30-0)^3 = 2160000 m^3$ = Vol of Water at begining

$A(0, 2160000)$

B is where cuts t axis $\Rightarrow V = 0$

$0 = 80(30-t)^3$ $(\div 80)$

$0 = (30-t)^3$ ($\sqrt[3]{}$ each side)

$0 = 30 - t$ $B(30, 0)$

$t = 30$ mins = time to empty water.

(ii) Water (Vol) after 10 mins

$V = 80(30-10)^3$

$= 80(20)^3 = 640000 m^3$

(iv) find $\frac{dV}{dt}$ when $t = 10$.

$V = 80(30-t)^3$

$\frac{dV}{dt} = 240(30-t)^2(-1)$

$= -240(30-t)^2$

at $10 = t$

$\frac{dV}{dt} = -240(30-10)^2$

$= -96000 m^3/min$

Water drains out at $96000 m^3/min$

(iii) $T = 0$ $V = 2160000$

$T = 10$ $V = 640000$

Average Value

$= \frac{2160000 - 640000}{10 - 0}$

$= \frac{1520000}{10}$

$= 152000 m^3/min.$

Q10 $\quad y = \dfrac{x}{\sqrt{1-x^2}}$

$y = \dfrac{x}{(1-x^2)^{\frac{1}{2}}}$ Quotient Rule.

$$\frac{dy}{dx} = \frac{(1-x^2)^{\frac{1}{2}}(1) + (x)\left(\frac{1}{\cancel{2}}(1-x^2)^{-\frac{1}{2}}(-\cancel{2}x)\right)}{\left[(1-x^2)^{\frac{1}{2}}\right]^2}$$

$$= \frac{(1-x^2)^{\frac{1}{2}} - (x^2)(1-x^2)^{-\frac{1}{2}}}{(1-x^2)^1}$$

$$= \frac{\sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}}}{1-x^2}$$ ($\leftarrow$ fractions $\Rightarrow$ C.D.)

$$= \frac{\dfrac{1-x^2-x^2}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{\dfrac{1}{\sqrt{1-x^2}}}{1-x^2}$$ (flippin fun)

$$= \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}$$

$$= \frac{1}{(1-x^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-x^2)^1}$$ (mult $\Rightarrow$ add powers)

$$= \frac{1}{(1-x^2)^{3/2}}$$ $\therefore k = 3/2$.

Q11

$y = \tan^{-1}\left(\frac{1}{x}\right)$ find $\frac{dy}{dx}$ at $x=1$.

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot -x^{-2}$$

$$\left[\begin{array}{l} \frac{1}{x} = x^{-1} \\ \frac{dy}{dx} = -x^{-2} \end{array}\right]$$

$$= \frac{-1}{\left[1+\left(\frac{1}{x}\right)^2\right]x^2}$$

at $x=1$

$$\frac{dy}{dx} = \frac{-1}{\left[1+\left(\frac{1}{1}\right)^2\right](1)^2} = \frac{-1}{2} = -\frac{1}{2}.$$

Q12 eqn Tangent: $y - y_1 = m(x - x_1)$ $\boxed{y = x^3 e^x}$

Point $x=0 \Rightarrow y = (0)^3 e^0 = (0)(1) = 0$ $(0,0)$

Slope $= \frac{dy}{dx}$ (product rule)

$$\frac{dy}{dx} = x^3 e^x + e^x 3x^2$$

at $x=0 \Rightarrow 0^3 e^0 + e^0 3(0)^2$

$0(1) + (1)3(0)$

$0 + 0 = 0$ slope

eqn of Tangent.

$y - 0 = 0(x-0)$

$\underline{\underline{y = 0}}$

Q13 $y = kx^2$

$\frac{dy}{dx} = 2kx$

$$x \cdot \frac{dy}{dx} + \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + y = 0$$

$$x(2kx) + \frac{1}{2}(2kx)^2 + kx^2 = 0$$

$$2kx^2 + \frac{1}{2}\left[4k^2x^2\right] + kx^2 = 0$$

$$3kx^2 + 2k^2x^2 = 0 \quad (\div x^2)$$

$$3k + 2k^2 = 0 \quad (\text{factorise})$$

$$k(3+2k) = 0$$

$$k = 0 \qquad 2k = -3$$

$$k = -3/2$$

But $k \neq 0$ $\therefore$ Sol $k = -3/2$.

Q14 $f(x) = x^3 + x^2 - 1$ $\qquad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = 3x^2 + 2x$

$x_1 = 1$.

$f(x_1) = f(1) = (1)^3 + (1)^2 - 1 = 1$

$f'(x_1) = f'(1) = 3(1)^2 + 2(1) = 5$.

$\Rightarrow x_2 = 1 - \frac{1}{5}$

$x_2 = 4/5$.

Q15 $y = \ln(1+e^x)$ show $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x}$$

$$\frac{d^2y}{dx^2} = \frac{(1+e^x)(e^x) - (e^x)(e^x)}{(1+e^x)^2}$$

$$= \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2}$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$$

$$\frac{e^x}{(1+e^x)^2} + \left(\frac{e^x}{1+e^x}\right)^2 = \frac{e^x}{1+e^x}$$

$$\frac{e^x}{(1+e^x)^2} + \frac{e^{2x}}{(1+e^x)^2} = \frac{e^x}{1+e^x}$$

$$\frac{e^x + e^{2x}}{(1+e^x)^2} = \frac{e^x}{1+e^x}$$

$$\frac{e^x(1+e^x)}{(1+e^x)^2} = \frac{e^x}{1+e^x}$$

$$\frac{e^x}{1+e^x} = \frac{e^x}{1+e^x} \quad \text{QED.}$$

Q16 $y = x^3 - x + 1$

(i) At pt B $x = (-1+h)$ $\therefore y = (-1+h)^3 - (-1+h) + 1$

$= -1 + 3h - 3h^2 + h^3 + 1 - h + 1$

$= 1 + 2h - 3h^2 + h^3$

(ii) gradient $=$ slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$

Co-ords of A: at Pt A $x = -1$ $\therefore y = (-1)^3 - (-1) + 1$

$= -1 + 1 + 1$

$= 1$

$A(-1, 1)$.

$A(-1,1)$ $B(-1+h,\ 1+2h-3h^2+h^3)$

$$m = \frac{1+2h-3h^2+h^3 - 1}{-1+h-(-1)}$$

$$= \frac{2h-3h^2+h^3}{h}$$

$$= 2 - 3h + h^2$$

(iii) limit as $h \to 0$

$= 2 - 3(0) + (0)^2$

$= 2$.