Ex 2.7

Q1 (i) $y = \sin^{-1} 6x$ $\quad y = \sin^{-1} u$ $\quad u = 6x$, $\frac{du}{dx} = 6$

$$\frac{dy}{du} = \frac{1}{\sqrt{1^2 - u^2}} = \frac{1}{\sqrt{1-36x^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1-36x^2}} \cdot 6 = \frac{6}{\sqrt{1-36x^2}}$$

(ii) $y = \tan^{-1} 3x$ $\quad y = \tan^{-1} u$ $\quad u = 3x$, $du/dx = 3$

$$\frac{dy}{du} = \frac{1}{1^2 + u^2} = \frac{1}{1+9x^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{1+9x^2} \cdot 3 = \frac{3}{1+9x^2}$$

(iii) $y = \sin^{-1}(2x+1)$ $\quad u = 2x+1$

$y = \sin^{-1} u$ $\quad du/dx = 2$

$$dy/du = \frac{1}{\sqrt{1^2-u^2}} = \frac{1}{\sqrt{1-(2x+1)^2}} = \frac{1}{\sqrt{1-4x^2-4x-1}} = \frac{1}{\sqrt{-4x^2-4x}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{-4x^2-4x}} \cdot 2 = \frac{2}{\sqrt{-4x^2-4x}}$$

(iv) $\tan^{-1}(x^2)$ $\quad u = x^2$

$y = \tan^{-1} u$ $\quad du/dx = 2x$

$$\frac{dy}{du} = \frac{1}{1+u^2} = \frac{1}{1+x^4}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{1}{1+x^4} \cdot 2x = \frac{2x}{1+x^4}$$

Q2 $y = \sin^{-1}(3x-1)$

$u = 3x-1$

$du/dx = 3$

$y = \sin^{-1} u$

$$dy/du = \frac{1}{\sqrt{1^2-u^2}} = \frac{1}{\sqrt{1^2-(3x-1)^2}} = \frac{1}{\sqrt{1-9x^2+6x-1}}$$

$$= \frac{1}{\sqrt{-9x^2+6x}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{-9x^2+6x}} \cdot 3 = \frac{3}{\sqrt{-9x^2+6x}}$$

Q3

(i) $\sin^{-1} 2x$ at $x = 0$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1^2-4x^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

at $x = 0 \Rightarrow \frac{2}{\sqrt{1-4(0)^2}} = \frac{2}{\sqrt{1}} = 2.$

(ii) $y = \tan^{-1} 4x$ at $x = 1/4$.

$$\frac{dy}{dx} = \frac{1}{1+16x^2} \cdot 4 = \frac{4}{1+16x^2}$$

at $x = \frac{1}{4} \Rightarrow \frac{4}{1+16(\frac{1}{4})^2} = \frac{4}{1+1} = \frac{4}{2} = 2.$

Q4 (i) $f(x) = \sin^{-1} \frac{3}{x}$

$f(x) = \sin^{-1} 3x^{-1}$

$u = 3x^{-1}$

$\frac{du}{dx} = -3x^{-2} = \frac{-3}{x^2}$

$y = \sin^{-1} u.$

$$\frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1-(3/x)^2}} = \frac{1}{\sqrt{1-9/x^2}} = \frac{1}{\sqrt{\frac{x^2-9}{x^2}}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{1}{\sqrt{\frac{x^2-9}{x^2}}} \cdot \frac{-3}{x^2}$$

$$= \frac{1}{\frac{\sqrt{x^2-9}}{x}} \cdot \frac{-3}{x^2} = \frac{-3}{x^2 \cdot \frac{\sqrt{x^2-9}}{x}}$$

$$= \frac{-3}{x\sqrt{x^2-9}}$$

(ii) $f(x) = \tan^{-1} \frac{x}{4}$

$y = \tan^{-1} u.$

$u = \frac{x}{4}$

$\frac{du}{dx} = \frac{1}{4}$

$$\frac{dy}{du} = \frac{1}{1+u^2} = \frac{1}{1+x^2/16} = \frac{1}{\frac{16+x^2}{16}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\frac{16+x^2}{16}} \cdot \frac{1}{4}$$

$$= \frac{1}{4\left(\frac{16+x^2}{16}\right)} = \frac{1}{\frac{16+x^2}{4}} = \frac{4}{x^2+16}.$$

Q5 (i) $y = x \sin^{-1} x$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1}x\,(1)$$

$$= \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$$

(ii) $y = 2x \operatorname{Tan}^{-1} x$

$$\frac{dy}{dx} = 2x\left(\frac{1}{1^2+x^2}\right) + \left(\operatorname{Tan}^{-1}x\right)(2)$$

$$= \frac{2x}{1+x^2} + 2\tan^{-1}x$$

Q6

$$y = \left(\sin^{-1}x\right)^2$$

$$\frac{dy}{dx} = 2\sin^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

Q7 $f(x) = \sin^{-1}(\cos x)$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$y = \sin^{-1} u.$

$$\frac{dy}{du} = \frac{1}{\sqrt{1^2 - u^2}} = \frac{1}{\sqrt{1^2 - (\cos x)^2}} = \frac{1}{\sqrt{1 - \cos^2 x}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\sqrt{1 - \cos^2 x}} \cdot -\sin x$$

$$\frac{1}{\sqrt{\sin^2 x}} \cdot -\sin x = \frac{-\sin x}{\sin x} = -1$$

$K = -1$

Q8 $f(x) = \tan^{-1}(\cos x)$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$y = \tan^{-1} u$

$$\frac{dy}{du} = \frac{1}{1 + u^2} = \frac{1}{1 + \cos^2 x}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{\cancel{1} + \cos^2 x} \cdot -\sin x$$

$$= \frac{-\sin x}{1 + \cos^2 x}$$

at $x = \pi/6$ $\quad \frac{-\sin \pi/6}{1 + \cos^2 \pi/6} = \frac{-\frac{1}{2}}{1 + (\sqrt{3}/2)^2}$

$$\frac{-1/2}{1 + 3/4} = \frac{-\frac{1}{2}}{\frac{7}{4}} = -\frac{1}{2} \cdot \frac{4}{7} = \frac{-4}{14} = \frac{-2}{7}$$

Q9 $y = \tan^{-1}\left(\frac{1}{x}\right)$ at $x = 1$

$y = \tan^{-1}(x^{-1})$ $u = x^{-1}$

$\frac{du}{dx} = -1x^{-2} = \frac{-1}{x^2}$

$y = \tan^{-1} u$

$$\frac{dy}{du} = \frac{1}{1^2+u^2} = \frac{1}{1^2+(x^{-1})^2} = \frac{1}{1+\frac{1}{x^2}} = \frac{1}{\frac{x^2+1}{x^2}} = \frac{x^2}{x^2+1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{x^2}{x^2+1} \cdot \frac{-1}{x^2} = \frac{-x^2}{x^2(x^2+1)}$$

at $x = 1$ $\frac{-(1)^2}{(1)^2(1^2+1)} = \frac{-1}{2}$

Q10 $y = \tan^{-1}(3x^2)$ at $x = 1/3$

$u = 3x^2$

$y = \tan^{-1} u$ $\frac{du}{dx} = 6x$

$$\frac{dy}{du} = \frac{1}{1^2-u^2} = \frac{1}{1+(3x^2)^2}$$

$$= \frac{1}{1+9x^4}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \Rightarrow \frac{1}{1+9x^4} \cdot 6x$$

$= \frac{6x}{1+9x^4}$ at $x = \frac{1}{3}$

$$= \frac{6\left(\frac{1}{3}\right)}{1+9\left(\frac{1}{3}\right)^4} = \frac{2}{1+9\left(\frac{1}{81}\right)} = \frac{2}{1+\frac{1}{9}} = \frac{2}{\frac{10}{9}} = 2 \times \frac{9}{10}$$

$$= \frac{9}{5}$$

Q11 $y = \tan^{-1} x$.

Show $\frac{d^2y}{dx^2}(1+x^2) + 2x\frac{dy}{dx} = 0$.

$y = \tan^{-1} x$

$\frac{dy}{dx} = \frac{1}{1^2+x^2} = (1+x^2)^{-1}$

$\frac{d^2y}{dx^2} = -1(1+x^2)^{-2}(2x) = \frac{-2x}{(1+x^2)^2}$

Hence: $\frac{d^2y}{dx^2}.(1+x^2) + 2x\frac{dy}{dx} = 0$

$= \frac{-2x}{(1+x^2)^{\cancel{2}}}\cancel{(1+x^2)} + 2x.\frac{1}{(1+x^2)}$

$= \frac{-2x}{1+x^2} + \frac{2x}{1+x^2} = 0$ QED.