Solutions to Paper 1 – Sample A

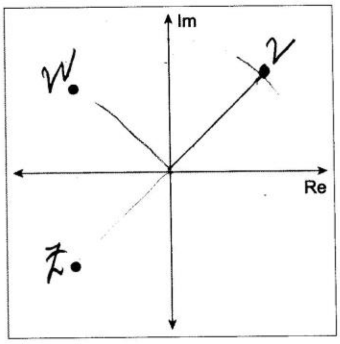
Section A **Concepts and Skills** **150 marks**

Answer all 6 questions in this section.

Question 1 **(25 marks)**

(a) Numbers w and z are plotted on an Argand diagram. Given that $z = iw$, where $i^2 = -1$, label the points on the diagram to show which point corresponds to which number.

Give a reason to support your answer.



(b) On the same diagram, show the position of number v where $v = -iw$.

Justify your answer.

(c) Give the range of k values, $k \in \mathbb{R}$, for which the quadratic equation $z^2 - kz + 2k = 0$ has non-real roots. $b^2 - 4ac < 0$

$(-k)^2 - 4(1)(2k) < 0$

$k^2 - 8k < 0$

$k(k-8) < 0$

$k = 0 \quad k = 8$

$0 < k < 8$

Question 2 **(25 marks)**

(a) Solve the equation $(1 + 4x)^2 > (1.)^2$

$1 + 8x + 16x^2 > 1$

$16x^2 + 8x > 1$

$8x(2x + 1) > 0$

$x = 0 \quad x = -1/2$

$x < -1/2 \quad x > 0$

Given $s = \log_b y$, express in terms of s,

4$\log_b by^2 + \log_b \sqrt{y} + \log_b (by)^2$

$4(\log_b b + 2\log_b y) + \frac{1}{2}\log_b y + 2[\log_b b + \log_b y]$

$4(1 + 2x) + \frac{1}{2}(x) + 2(1 + x)$

$4 + 8x + \frac{1}{2}x + 2 + 2x$

$10\frac{1}{2}x + 6$

3 **(25 marks)**

Solve the simultaneous equations

$4x + y - 2z = -1$

$x + 3y - z = 3$

$2x + 5y + 3z = 0$

A: $4x + y - \cancel{2z} = -1$
$-2B$: $-2x - 6y + \cancel{2z} = -6$

D: $2x - 5y = -7$

$3B$: $3x + 9y - \cancel{3z} = 9$
C: $2x + 5y + \cancel{3z} = 0$

E: $5x + 14y = 9$

$2E$: $\cancel{10x} + 28y = 18$
$-5D$: $-\cancel{10x} + 25y = 35$

$53y = 53$

$y = 1$

D: $2x - 5 = -7$

$2x = -2$

$x = -1$

B: $-1 + 3 - z = 3$

$-z = 1$

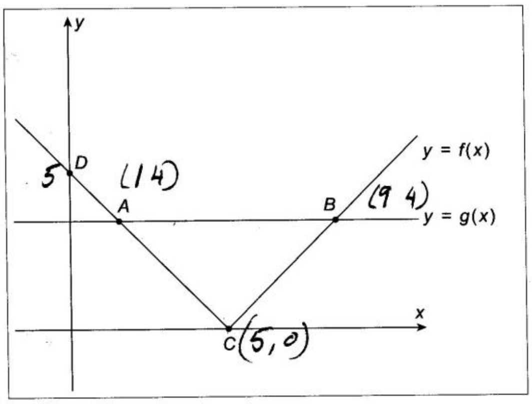
$z = -1$

$(-1, 1, -1)$

EDCO SAMPLE A PAPER 1

(b) The graphs of the functions $f: x \to |x - 5|$ and $g: x \to 4$ are shown in the diagram.

$y = x - 5$
$y = |-5|$
$x = 5$
$|x - 5| = \cdot 4$
$x = 9/1$



Find the coordinates of the points A, B, C and D.

$A = (\ 1\ ,\ 4\ )$	$B = (\ 9\ ,\ 4\ )$
$C = (\ 5\ ,\ 0\ )$	$D = (\ 0\ ,\ 5\ )$

Hence, or otherwise, solve $|x - 5| < 4$

$1 < x < 9$

Question 4 **(25 marks)**

(a) Write the recurring decimal 0.503 503 503... as an infinite geometric series and hence as a fraction.

$\frac{503}{1000} + \frac{503}{1000000} + \ldots$ $\quad a = \frac{503}{1000} \quad r = \frac{1}{1000}$

$S_\infty = \frac{a}{1-r} = \frac{503/1000}{1 - 1/1000}$

$= \frac{\frac{503}{1000}}{\frac{999}{1000}} = \frac{503}{1000} \times \frac{1000}{999}$

$= \frac{503}{999}$

(b) Prove, by induction, that

$$n! > 2^{n-1}, \text{ for } n \geq 3, n \in \mathbb{N}.$$

$n = 3$ $\quad 3! > 2^{3-1}$

$6 > 4$ True

$n = k$ $\quad k! > 2^{k-1}$

$n = k+1$ $\quad (k+1)! > 2^{k}$

$(k+1)! = k!(k+1)$

But $k! = 2^{k-1}$

$k! > 2^{k-1}$

$(k+1)k! > 2^{k-1}(k+1)$

$(k+1)! > 2^{k-1}(k+1)$

now prove $2^{k} > 2^{k-1}(k+1)$

$2^{k} > 2^{k}.2^{-1}(k+1)$

$1 > \frac{k+1}{2}$ since $k \geq 3$

$1 > 2$ True.

Question 5 **(25 marks)**

The closed interval $A = \{x \mid -2 \leq x \leq 2, x \in \mathbb{R}\}$. The function f is defined by $f: A \to \mathbb{R}: x \to 2x^3 + x^2 - 8x - 4$.

(a) Find the maximum and minimum values of f.

@ $x = -2$ $\quad f(x) = 2(-2)^3 + (-2)^2 - 8(-2) - 4 = 0$

@ $x = 2$ $\quad f(x) = 2(2)^3 + (2)^2 - 8(2) - 4 = 0$

$f'(x) = 6x^2 + 2x - 8 = 0$

$3x^2 + x - 4 = 0$

$(3x + 4)(x - 1) = 0$

$x = -4/3 \quad x = 1$

$f(-4/3) = 100/27$ max

$f(1) = -9$ min.

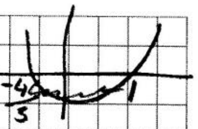
$d^2y/dx^2 \quad 12x + 2$ at $x = -4/3$ = Neg Max @ $x = 1$ = pos $\Rightarrow$ min.

(b) (i) For what values of x is the slope of the function always decreasing?

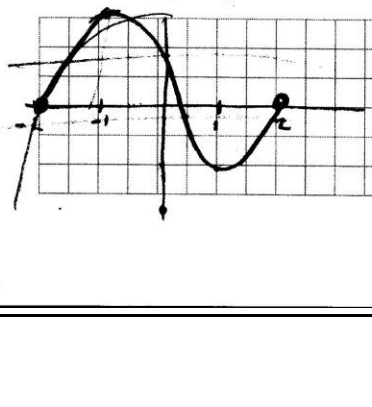
dy/dx is neg $\quad 6x^2 + 2x - 8 < 0$

$x = -4/3 \quad x = 1$

$-4/3 < x < 1$



(ii) State whether f is injective or not. Give a reason for your answer.



Horizontal line test cuts more than once $\Rightarrow$ not injective.

EDCO SAMPLE A PAPER 1

Question 6 **(25 marks)**

Given $f(x) = x + 2$, $g(x) = \dfrac{1}{x-2}$ and $h(x) = \dfrac{x^2-4}{x-2}$

(a) Fill in the table below.

x	1.9	1.99	2	2.01	2.1
$f(x)$	3.9	3.99	4	4.01	4.1
$g(x)$	−10	−100	—	100	10
$h(x)$	3.9	3.99	—	4.01	4.01

(b) From your answers in part (a), decide whether or not a limit exists as the value of x approaches 2 for $f(x)$, $g(x)$ and $h(x)$.

	Limit as $x \to 2$, ✓ or ✗	Reason for your answer
$f(x)$	yes ✓	both directions approaching 4 has a value @ $x=2$
$g(x)$	No ✗	No value + opp directions go in opp directions
$h(x)$	yes ✓	both directions approach 4.

(c) Decide whether or not $f(x)$, $g(x)$ and $h(x)$ are continuous as the value of x approaches 2, giving a reason for your answer in each case.

	Continuous, ✓ or ✗	Reason for your answer
$f(x)$	✓ yes	has a value @ $x=2$
$g(x)$	✗	no value @ $x=2$
$h(x)$	✗	no value @ $x=2$

Section B	Contexts and Applications	150 marks
Answer all three questions in this section.		

Question 7 **(50 marks)**

A ball is thrown upwards from the top of a building. During the motion, its velocity v, in m/s, is given by the equation

$$v = 15 - 10t \text{ where } t \text{ is the time in seconds.}$$

(a) Complete the table below, from the start ($t = 0$) for the next three seconds of the motion.

t	0	1	2	3
v	15	5	−5	−15

(b) Show that the average velocity over this period of the motion is 0 m/s.

$\frac{15 + 5 - 5 - 15}{4} = \frac{0}{4} = 0$ m/s

(c) Verify, using integration methods, that the average velocity over the same period is 0 m/s.

$\frac{1}{3-0}\int_0^3 (15-10t)\,dt = \frac{1}{3}\left[15t - 5t^2\right]_0^3$

$= \frac{1}{3}\left[\left(15(3) - 5(3)^2\right) - (0)\right] = \frac{1}{3}(0 - 0) = \frac{1}{3}0 = 0$ m/s

(d) After how many seconds will the average velocity be −3 m/s?

$\frac{1}{t}\left(15t - 5t^2\right)_0^t = -3$

$\frac{1}{t}\left[15t - 5t^2 - 0\right] = -3$

$15t - 5t^2 = -3t$

$18t - 5t^2 = 0$

$18 - 5t = 0$

$t = 18/5$ sec

$t = 3.6$ sec

(e) Show that the average velocity can never equal 20 m/s.

$\frac{1}{t}(15t - 5t^2) = 20$

$15t - 5t^2 = 20t$

$-5t^2 = 5t$

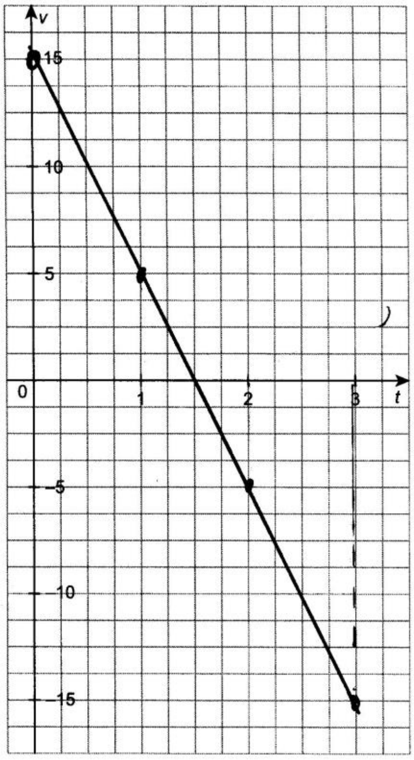
$-5t = 5$

$t = -1$

time cannot be neg.

EDCO SAMPLE A PAPER 1

(f) Draw a velocity–time graph for the closed interval $0 \leq t \leq 3$.



(g) From your graph, calculate the total area bounded by the velocity–time graph and the x-axis between $t = 0$ and $t = 3$.

Area $\Delta = \frac{1}{2}(1.5)(15) \quad \times 2$

$= \frac{45}{4} \times 2$

$= \frac{45}{2} = 22.5.$

(h) Using integration methods, verify the answer in part (g).

$\int_0^{1.5} (15 - 10t)\,dt = \left(15t - 5t^2\right)_0^{1.5}$

$\left(15(1.5) - 5(1.5)^2\right) - (0) =$

$\frac{45}{2} - \frac{45}{4} = \frac{45}{4} \times 2 = \frac{45}{2} = 22.5.$

(i) What does the area represent in this case? Distance.

(j) Calculate the deceleration during the motion. $v = 15 - 10t$

$$\frac{dv}{dt} = -10 \ m/s^2$$

Question 8 **(50 marks)**

Mary borrows €15,000 at an APR (Annual Percentage Rate) of 6%.

She takes out the loan over five years, and wishes to repay it in equal instalments.

Her first instalment is to be paid one year after she takes out the loan.

(a) If the amount of the first instalment is A, write down, in terms of A, the present values of the other instalments.

Instalment	Present value
1st	$A/1.06$
2nd	$A/1.06^2$
3rd	$A/1.06^3$
4th	$A/1.06^4$
5th	$A/1.06^5$

(b) The five instalments form a geometric series.

Use this fact to express the sum of the instalments in terms of A.

$$\frac{A}{1.06} + \frac{A}{(1.06)^2} + \frac{A}{(1.06)^3} + \dots \frac{A}{(1.06)^5}$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \qquad a = \frac{A}{1.06} \quad r = \frac{1}{1.06} \quad n = 5$$

$$\frac{A}{1.06}\left(\frac{1-(1/1.06)^5}{1-1/1.06}\right) = \text{Total}.$$

$$\frac{A}{1.06}(4.465105613) = \text{Total}$$

$$4.212363786\,A = \text{Total}.$$

EDCO SAMPLE A PAPER 1

(c) Find, to the nearest cent, the value of A that corresponds to the present loan of €15,000.

$$15000 = 4.212363786\,A$$

$$3560.95 = A$$

(d) Verify your answer using the general formula for an amortised loan (Mathematical Tables p.31).

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1}$$

$$A = \frac{15000\,(0.06)(1+0.06)^5}{(1+0.06)^5 - 1}$$

$$A = \cancel{\text{[illegible]}}\ 3560.95$$

(c) Mary decided to pay a total of 60 monthly instalments instead of 5 yearly instalments. The APR remains at 6%. She pays the first instalment one month after she takes out the loan. By letting the monthly instalment = A,

(i) What is the present value of the 1st, 2nd, and 60th instalments?

1st instalment	$\frac{A}{(1\cdot06)^{1/12}}$
2nd instalment	$\frac{A}{(1\cdot06)^{2/12}}$
60th instalment	$\frac{A}{(1\cdot06)^{60/12}}$

(ii) Find the value of A, the monthly instalment.

$$15000 = \frac{A}{(1\cdot06)^{1/12}} + \frac{A}{(1\cdot06)^{2/12}} + \cdots \frac{A}{(1\cdot06)^{60/12}}$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad a = \frac{A}{(1\cdot06)^{1/12}} \quad r = \frac{1}{(1\cdot06)^{1/12}} \quad n=60$$

$$15000 = \frac{\frac{A}{(1\cdot06)^{1/12}}\left(1-\left(\frac{1}{(1\cdot06)^{1/12}}\right)^{60}\right)}{1-\frac{1}{(1\cdot06)^{1/12}}}$$

$$15000 = \frac{A}{(1\cdot06)^{1/12}}(52\cdot17656342)$$

$$15000 = A(51\cdot92382159)$$

$$288\cdot88 = A$$

Question 9 $\frac{dV}{dt}$ **(50 marks)**

(a) A spherical football is pumped with air at a rate of 25 cm^3 per second. Calculate the rate at which the radius of the football is increasing when the radius is 10 cm. Give your answer in terms of π.

$\frac{dr}{dt}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$\frac{dr}{dt} = 25 \times \frac{1}{4\pi r^2}$

@ $r = 10$

$25 \times \frac{1}{4\pi(10)^2}$

$= \frac{1}{16\pi}$

(b) Differentiate $\sin^2(3x - 1)$ with respect to x.

$y = \sin^2(3x-1)$

$\frac{dy}{dx} = 2\sin(3x-1)\cos(3x-1)(3)$

$= 6\sin(3x-1)\cos(3x-1)$

or $\frac{dy}{dx} = \underline{2\sin(3x-1)\cos(3x-1)}(3)$ Tables p.14

$= \sin 2(3x-1)(3)$

$= 3\sin 2(3x-1)$

(c) The function $f(x) = \frac{2}{1-2x}$ is defined for $x \in \mathbb{R}$, $x \neq \frac{1}{2}$.

(i) Show that $f(x)$ is always increasing and has no points of inflection.

$f'(x)$ is pos.

$f'(x) = \frac{(1-2x)(0) - (2)(-2)}{(1-2x)^2}$

$\frac{4}{(1-2x)^2} > 0$. is always pos as denominator sq'ed is positive.

$f''(x) = \frac{(1-2x)^2(0) - 4(2)(1-2x)(-2)}{(1-2x)^4}$

$\frac{16-32x}{(1-2x)^4} \neq 0$.

$16 - 32x = 0$

$32x = 16$

$x = \frac{1}{2} \neq 0$.

(ii) The line $y = x + c$ is a tangent to the graph of $f(x)$.

$m = 1$

Find the two possible values of c, $c \in \mathbb{R}$.

$\Rightarrow f'(x) = 1$

$$\frac{4}{(1-2x)^2} = 1$$

$$4 = (1-2x)^2$$

$$4 = 1 - 4x + 4x^2$$

$$4x^2 - 4x - 3 = 0$$

$$(2x - 3)(2x + 1) = 0$$

$$x = 3/2 \quad x = -1/2$$

$$y = \frac{2}{1-2x}$$

$$@\ x = 3/2 \quad y = \frac{2}{1-2(3/2)} = -1$$

$$(3/2, -1)$$

$$y = x + c$$

$$-1 = 3/2 + c$$

$$-5/2 = c$$

$$@\ x = -1/2$$

$$y = \frac{2}{2-2(1/2)} = 1$$

$$(-1/2, 1)$$

$$= -1/2 + c$$

(d) The function $f(x) = \dfrac{2x}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$.

(i) A and B are two different points on the curve. The tangents at points A and B are parallel.

If the coordinates of A are $(-1, 1)$, find the coordinates of B.

$$f'(x) = \frac{(x-1)(2) - (2x)(1)}{(x-1)^2} = \frac{2x - 2 - 2x}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

$$\text{at } (-1, 1) \quad f'(x) = \frac{-2}{(-1-1)^2} = \frac{-2}{4} = -1/2.$$

$$y = -1/2\, x + c \quad (-1, 1) \quad 1 = -1/2 + c \quad c = 3/2. \quad \text{Eqn } 1 = y = -1/2\, x + 3/2$$

at B tangent is parallel. $\Rightarrow$ same slope

$$\frac{-2}{(x-1)^2} = -1/2.$$

$$-4 = -(x-1)^2$$

$$-4 = -x^2 + 2x - 1$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \quad x = -1$$

pt A.

at $x = 3$

$$y = \frac{2(3)}{3-1} = \frac{6}{2} = 3$$

$$B(3, 3).$$

EDCO SAMPLE A
PAPER 1

(ii) Write the equations of the asymptotes to the curve of $f(x)$.

Hence, show that the mid-point of [A, B] is the point of intersection of the asymptotes.

$f(x) = \frac{2x}{x-1}$ @ $x=1$ no value

$\Rightarrow x=1$ is one asymptote

$\lim_{x \to \infty} \frac{2x}{x-1} = \lim_{x \to \infty} \frac{2x/x}{x/x - 1/x} = \frac{2}{1-\frac{1}{\infty}} = 2$

$y=2$ is 2^{nd} asymptote

Mid pt. $A(-1,1)$ $B(3,3)$

Mid pt $\left(\frac{-1+3}{2}, \frac{1+3}{2}\right)$

$(1, 2)$.