Appendix: Trigonometric Formulae

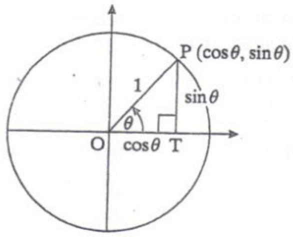
It will be assumed that these formulae are established in the order listed here. In deriving any formula, use may be made of formulae that precede it.

1. $\cos^2 A + \sin^2 A = 1$
2. sine formula: $\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$
3. cosine formula: $a^2 = b^2 + c^2 - 2\,bc \cos A$
4. $\cos (A-B) = \cos A \cos B + \sin A \sin B$
5. $\cos (A+B) = \cos A \cos B - \sin A \sin B$
6. $\cos 2A = \cos^2 A - \sin^2 A$
7. $\sin (A+B) = \sin A \cos B + \cos A \sin B$
8. $\sin (A-B) = \sin A \cos B - \cos A \sin B$
9. $\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
10. $\tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
11. $\sin 2A = 2 \sin A \cos A$
12. $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$
13. $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$
14. $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$
15. $\cos^2 A = \frac{1}{2} (1 + \cos 2A)$
16. $\sin^2 A = \frac{1}{2} (1 - \cos 2A)$
17. $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$
18. $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$
19. $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$
20. $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$
21. $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
22. $\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$
23. $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
24. $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

Formal Proofs of Trigonometric Formulae Required for L.C Higher Level

1. $\cos^2\theta + \sin^2\theta = 1$

We have already established that any point on the **unit circle** is defined by the coordinates **($\cos\theta$, $\sin\theta$)**.



In the given diagram $|OP| = 1$

$\Rightarrow \quad |OP|^2 = 1$

$\Rightarrow \quad \sqrt{(\cos\theta - 0)^2 + (\sin\theta - 0)^2} = 1$

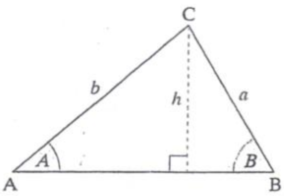
$\Rightarrow \quad \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\Rightarrow \quad \mathbf{\cos^2\theta + \sin^2\theta = 1}$ ... (squaring both sides) $*$

2. Sine Formula: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Proof

Construct a perpendicular h from C to [AB].



(i) $\dfrac{h}{b} = \sin A \quad \Rightarrow \quad h = b\sin A$

(ii) $\dfrac{h}{a} = \sin B \quad \Rightarrow \quad h = a\sin B$

$\Rightarrow \quad a\sin B = b\sin A$

Divide both sides by $\sin A \sin B$

$\Rightarrow \quad \dfrac{a\sin B}{\sin A \sin B} = \dfrac{b\sin A}{\sin A \sin B}$

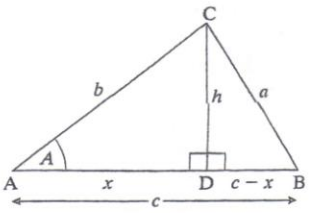
$\Rightarrow \quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

Similarly it may be shown that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Sine Rule $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ or $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

3. Cosine Formula: $a^2 = b^2 + c^2 - 2bc\,cosA$

***Proof**



In the $\triangle$ABC, CD is perpendicular to AB.

Let $|CD| = h$ and $|AD| = x$

Thus $|DB| = c - x$

We now apply the Theorem of Pythagoras to the triangles ACD and CDB.

In $\triangle$ACD: $h^2 + x^2 = b^2 \Rightarrow h^2 = b^2 - x^2$

In $\triangle$BCD: $h^2 + (c - x)^2 = a^2 \Rightarrow h^2 = a^2 - (c - x)^2$

Equating the two values for h^2, we get

$$a^2 - (c - x)^2 = b^2 - x^2$$

$$\Rightarrow a^2 - (c^2 - 2cx + x^2) = b^2 - x^2$$

$$\Rightarrow a^2 - c^2 + 2cx - x^2 = b^2 - x^2$$

$$\Rightarrow a^2 = b^2 + c^2 - 2cx \qquad \text{But } \frac{x}{b} = \cos A$$

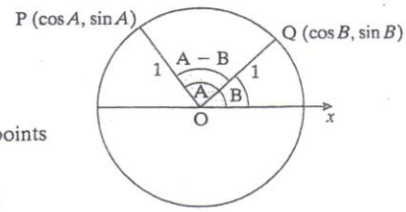
$$\Rightarrow a^2 = b^2 + c^2 - 2c\,(b \cos A) \qquad \Rightarrow x = b \cos A$$

$$\Rightarrow a^2 = b^2 + c^2 - 2bc \cos A$$

Similarly is may be proved that

$b^2 = c^2 + a^2 - 2ca \cos B$ and $c^2 = a^2 + b^2 - 2ab \cos C$

4. Cos (A – B) = cosAcosB + sinAsinB



Let the radii [OP] and [OQ] make angles A and B with the positive x-axis.

$|\angle POQ| = A - B$

We now find |PQ| using two different methods:

1. the standard formula for the distance between two points
2. the cosine rule.

1. Using the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, we have:

$$|PQ|^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= \cos^2 A - 2 \cos A \cos B + \cos^2 B + \sin^2 A - 2 \sin A \sin B + \sin^2 B$$

But $\sin^2 A + \cos^2 A = 1$ and $\sin^2 B + \cos^2 B = 1$.

$$\Rightarrow |PQ|^2 = 1 - 2 \cos A \cos B + 1 - 2 \sin A \sin B$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B) \ldots ①$$

2. Now using the cosine rule to find |PQ|, we have:

$$|PQ|^2 = |OP|^2 + |OQ|^2 - 2|OP|\,|OQ| \cos(A - B)$$
$$= 1 + 1 - 2(1)(1) \cos(A - B) \ldots |OP| = |OQ| = 1 = \text{radius}$$
$$= 2 - 2 \cos(A - B) \ldots ②$$

Equating the two values for $|PQ|^2$, we have

$$2 - 2(\cos A \cos B + \sin A \sin B) = 2 - 2 \cos (A - B)$$

$$\Rightarrow -(\cos A \cos B + \sin A \sin B) = -\cos(A - B)$$

*$\Rightarrow$ **$\cos(A - B) = \cos A \cos B + \sin A \sin B$ …… (i)**

5. **Cos (A + B) = cosAcosB – sinAsinB**

To derive the formula for $\cos(A+B)$, we replace B with $(-B)$ in formula **(i)** on the previous page:

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\Rightarrow \cos[A-(-B)] = \cos A \cos(-B) + \sin A \sin(-B)$$

$$= \cos A \cos B + \sin A(-\sin B)$$

$$*\Rightarrow \mathbf{\cos(A+B) = \cos A \cos B - \sin A \sin B} \text{ ...... (ii)}$$

6. **Cos 2A =** $\cos^2 A - \sin^2 A$

$$\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

7. **Sin (A + B) = sinAcosB + cosAsinB**

$\cos(A-B) = \cos A \cos B + \sin A \sin B$ ... from **(i)**

To derive the formula for $\sin(A+B)$, we replace A with $(90° - A)$.

$$\Rightarrow \cos(A-B) = \cos[(90°-A)-B] = \cos(90°-A)\cos B + \sin(90°-A)\sin B$$

$$= \sin A \cos B + \cos A \sin B$$

$$\Rightarrow \cos[90° - (A+B)] = \sin A \cos B + \cos A \sin B$$

$$*\Rightarrow \mathbf{\sin(A+B) = \sin A \cos B + \cos A \sin B} \text{ ...... (iii)}$$

Substituting $(-B)$ for B in formula **(iii)**, we get

$$\sin(A-B) = \sin A \cos(-B) + \cos A \sin(-B)$$

$$*\Rightarrow \mathbf{\sin(A-B) = \sin A \cos B - \cos A \sin B} \text{ ...... (iv)}$$

$\cos(-B) = \cos B$
$\sin(-B) = -\sin B$
$\sin(90° - A) = \cos A$
$\cos(90° - A) = \sin A$

8. **Tan(A + B) =** $\dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

We now divide each term in the numerator and denominator by $\cos A \cos B$.

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}}$$

$$*\Rightarrow \mathbf{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}} \text{ ...... (v)}$$